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FAR FIELD CONDITIONS IN STEADY FLOWS WITH A FREE STREAM MACH NUMBER ONE

K. G. GUDERLEY

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300 College Park Avenue
Dayton, Ohio 45469

June 1981
Interim Technical Report for period April - November 1980

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The far field in a steady flow with a free stream Mach number one is governed by the potential equation simplified by the assumption that the deviations from a sonic parallel flow are small. In a suitable system of coordinates the dominant expression has the form of a similarity solution. To formulate far field conditions (conditions imposed by far field to the computed part of the flow field at its outer edge) one considers perturbations to the

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dominant part of the distant field. The particular solutions of the linear partial differential equation so obtained have also similarity form. The far field conditions express the requirement that only such perturbations are admitted which die out more quickly than the basic flow as one moves toward infinity. The derivation uses a Laplace form technique; the absence of nonadmissible particular solutions expresses itself by the absence of certain poles of the Laplace transform. These conditions are ultimately formulated in terms of quantities available in the physical plane. One obtains integrals extended over the outer contour of the computed flow field which contain the potential and its gradient.

PREFACE

The work was performed during the period April through November 1980 under Grant AFOSR-78-3524 to the University of Dayton for the Applied Mathematics Group, Analysis and Optimization Branch, Structures and Dynamics Division, Flight Dynamics Laboratories under Project 2304, Task 2304N1, Work Unit 2304N110 and Program Element 61102F. Dr. Karl G. Guderley, of the University of Dayton Research Institute, was Principal Investigator. Dr. Charles L. Keller AFWAL/FIBRD (513) 255-5350, Flight Dynamic Laboratories, Wright-Patterson AFB, OH, 45433 was program manager.

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SECTION I
INTRODUCTION

In formulating far field conditions for steady subsonic flows one usually assumes that at a sufficient distance from the body the flow equations can be approximated by the potential equation linearized for the vicinity of a parallel flow with the assigned free stream Mach number. If the free stream Mach number is increased, then the supersonic region which arises at the profile extends and the boundary at which one is justified in applying far field conditions of this kind moves to larger and larger distances. In the limiting case where the free stream Mach number is one this boundary would lie at infinity. The present article derives far field conditions which are applicable if the free stream Mach number is one.

Far field conditions cannot be formulated unless one possesses analytical solutions by which the far field can be represented. In a flow with the free stream Mach number one, the basic field is a parallel flow with the sonic velocity and a superimposed expression which describes the dominant effect caused by the presence of a body. This dominant term satisfies a nonlinear partial differential equation, but ultimately it is given by one expression in the form of a product hypothesis. Certain parameters which one encounters in this expression (and which depend upon the size of the body) must be adjusted as the flow computations progress. To this basic field perturbations are superimposed which are sufficiently small so that they can be computed from linearized equations. Some perturbations of this kind increase faster as one goes to infinity than the basic "dominant" term described above. The far field conditions express the requirement that such perturbations are not admissible.

The investigations are carried out in the physical plane, even though the hodograph for plane flows is governed by linear equations. In the physical plane two and three dimensional problems can be treated in nearly the same manner. An important aid, especially in the three dimensional problem is the fact, that the solutions of the linearized equations can be expressed in closed form. (Refs. 1, 2, and 3). The author found the approach of Randall particularly useful.

Using coordinates suggested by the similarity solution for the basic flow one can map the field upstream of the limiting characteristic into a strip of finite width in an auxiliary plane. In this plane perturbations can be treated by means of a Laplace transform. (This is the technique of Reference 4.) The postulate that perturbations do not increase faster than the basic flow as one goes to infinity, in the physical plane, leads to the requirement that the poles in the right half of the plane for the Laplace transform vanish. From this requirement the far field conditions are derived. The behavior of the Laplace transform at some of the other poles is used to adjust the expression for the basic singularity. Ultimately, these conditions can be evaluated in the physical plane. Some flexibility in their application is obtained by means of Green's theorem. The report derives the underlying theory and the formulae needed in the practical application. The results hold for plane as well as for three-dimensional flows.

SECTION II
PARTIAL DIFFERENTIAL EQUATIONS

We consider the flow over a body at a free stream Mach number one. Let Φ be the potential that describes the deviation of such a flow from a parallel flow with the free stream Mach number one. At a sufficiently large distance from the body this deviation is small and one can use a simplified form of the potential equation. In a system of Cartesian coordinates x, y, z in which the x axis has the free stream direction, this partial differential equation is

$$-\left(\frac{\gamma+1}{2}\right)\frac{\partial}{\partial x}\left(\Phi_x^2\right) + \Phi_{yy} + \Phi_{zz} = 0 \tag{1}$$

where γ is the ratio of the specific heats. In cylindrical coordinates x, r, θ with $y = r\cos\theta$, and $z = r\sin\theta$, it assumes the form

$$-\left(\frac{\gamma+1}{2}\right)\frac{\partial}{\partial x}\left(\Phi_x^2\right) + \Phi_{rr} + \frac{1}{r}\Phi_r + \frac{1}{r^2}\Phi_{\theta\theta} = 0 \tag{2}$$

Let Φ_o be the part of the potential which is dominant at large distances. In the three dimensional case it is axisymmetric and therefore satisfies

$$-\frac{\gamma+1}{2}\frac{\partial}{\partial x}\left(\Phi_{o,x}\right)^2 + \Phi_{o,rr} + \frac{1}{r}\Phi_{o,r} = 0 \tag{3}$$

The corresponding equation for plane flow reads

$$-\frac{\gamma+1}{2}\frac{\partial}{\partial x}\left(\Phi_{o,x}\right)^2 + \Phi_{o,yy} = 0 \tag{4}$$

The Eqns. (3) and (4) may be written as a single equation

$$-\frac{\gamma+1}{2}\frac{\partial}{\partial x}\left(\Phi_{o,x}\right)^2 + \frac{\alpha}{\eta}\Phi_{o,\eta} + \Phi_{o,\eta\eta} = 0 \tag{5}$$

where respectively for plane and axisymmetric flows

$$\alpha = 0, \quad \eta = y \quad \text{and } \alpha = 1, \ \eta = r.$$

The solution Φ_o has the form

$$\Phi_o = \eta^{3n-2} f(\zeta) \tag{6}$$

where

$$\zeta = (\gamma+1)^{-1/3} x \eta^{-n} \tag{7}$$

The constant exponent n must be chosen in such a manner that certain boundary conditions are satisfied. One arrives at the familiar equation

$$(n^2\zeta^2 - f') f'' + (-5n+5-\alpha)n\zeta f' + (3n-2)(3n-3+\alpha)f = 0 \tag{8}$$

The lines $\zeta = -\infty$ and $\zeta = 0$ are respectively the negative x axis and the positive y axis. At the sonic line one has $\Phi_{o,x} = 0$ therefore it is given by $f' = 0$. The coefficient of f'', viz. $(n^2\zeta^2 - f')$ vanishes for a value of ζ for which the generalized parabola $\zeta=(\gamma+1)^{-1/3}xy^{-n}$ happens to have the characteristic direction. This is seen in the following manner. The characteristics of Eq. (5) have the directions

$$dx/d\eta = \pm(\gamma+1)^{1/2}\Phi_{o,x}{}^{1/2}$$

It follows from Eq. (7), that for constant ζ

$$dx/d\eta = +(\gamma+1)^{1/3} n\zeta\eta^{n-1}$$

If

$$f' - n^2\zeta^2 = 0$$

then

$$\zeta = n^{-1}(f')^{1/2}$$

One obtains from Eqs. (6) and (7)

$$\Phi_{o,x} = (\gamma+1)^{-1/3}\eta^{2n-2}f'(\zeta)$$

This leads to the above result. If $(n^2\zeta^2 - f') = 0$ and f'' is bounded, then the remaining terms in the differential equation must vanish. This gives the compatibility conditions for this characteristic. The line ζ = const for which this happens is called the limiting characteristic. All characteristics starting at the body upstream of the limiting characteristic end at the sonic line, those starting downstream do not impinge on the subsonic region.

The function f must satisfy the following conditions; along the line where $(n^2\zeta^2 - f') = 0$, f'' must remain bounded, the solution Φ_o must be symmetric with respect to the negative x axis in the two dimensional case ($\alpha = 0$), and it must be free of singularities in the axisymmetric case ($\alpha=1$). Expressions for f are availabe in a closed form. For the axisymmetric case we shall use the formulation given in Ref. 3. One finds

$$\begin{aligned} n &= 4/5 \quad \text{for} \quad \alpha = 0 \\ n &= 4/7 \quad \text{for} \quad \alpha = 1 \end{aligned} \tag{9}$$

The functions $f(\zeta)$ form a one parameter family. If one solution is given by

$$f(\zeta) = \bar{f}(\zeta)$$

then others are given by

$$f(\zeta) = \mu^3 \bar{f}(\bar{\zeta}) \tag{10}$$

with

$$\bar{\zeta} = \mu^{-1}\zeta$$

where μ is an arbitrary positive constant.

Perturbations to this solution are introduced by setting

$$\Phi = \Phi_o + \phi \tag{11}$$

and considering ϕ as small enough so that terms of the second order can be disregarded. This leads to the following linear partial differential equation

$$-(\gamma+1) \frac{\partial}{\partial x} (\Phi_{o,x}\, \phi_x) + \phi_{yy} + \phi_{zz} = 0 \tag{12}$$

or in a form which combines plane flows and three-dimensional flows in cylindrical coordinates

$$-(\gamma+1) \frac{\partial}{\partial x} (\eta^\alpha \Phi_{o,x}\phi_x) + \frac{\partial}{\partial \eta} (\eta^\alpha \phi_\eta) + \alpha \frac{\partial}{\partial \theta} (\eta^{\alpha-2}\phi_\theta) = 0 \tag{13}$$

Notice that the linear operator acting in this equation has a divergence form. The basic flow is expressed in terms of ζ and η. This suggests that these independent variables be used also in the equations for the perturbations. We set accordingly

$$\phi(x,\eta,\theta) = \tilde{\phi}(\eta,\zeta,\theta) \tag{14}$$

Substituting Eqs. (6) and (14) into Eq. (13) one obtains

$$(f' - n^2\zeta^2)\tilde{\phi}_{\zeta\zeta} + [f'' + (-n^2-n + \alpha n)\zeta]\tilde{\phi}_\zeta - \alpha\eta\tilde{\phi}_\eta + 2n\zeta\eta\tilde{\phi}_{\eta\zeta}$$
$$- \eta^2\tilde{\phi}_{\eta\eta} - \alpha\tilde{\phi}_{\theta\theta} = 0 \tag{15}$$

In a first step toward bringing this equation into a divergence form we eliminate the mixed second derivative by setting

$$\tilde{\phi}(\eta,\zeta,\theta) = \bar{\phi}(\rho,\zeta,\theta) \tag{16}$$

with

$$\rho = \eta\ h(\zeta) \tag{17}$$

With

$$\frac{h'}{h} = -\frac{n\zeta}{f'-n^2\zeta^2} \tag{18}$$

one obtains

$$(f'-n^2\zeta^2)\ \bar{\phi}_{\zeta\zeta} + [f" + (-n^2-n+\alpha n)\zeta]\bar{\phi}_\zeta$$

$$-\frac{f'}{f'-n^2\zeta^2}\ [\rho^2\bar{\phi}_{\rho\rho} + (n+\alpha)\rho\bar{\phi}_\rho] - \alpha\bar{\phi}_{\theta\theta} = 0$$

(19)

Using Eq. (18) one derives (or verifies) the following divergence form of this equation

$$\frac{\partial}{\partial\zeta}[\frac{\rho^{n+\alpha-2}(f'-n^2\zeta^2)}{h^{n+\alpha-1}}\ \bar{\phi}_\zeta(\rho,\zeta,\theta)] - \frac{\partial}{\partial\rho}[\frac{f'\ \rho^{n+\alpha}}{(f'-n^2\zeta^2)h^{n+\alpha-1}}\bar{\phi}_\rho(\rho,\zeta,\theta)]$$

$$-\ \alpha\ \frac{\partial}{\partial\theta}[\frac{\rho^{n+\alpha-2}}{h^{n+\alpha-1}}\ \bar{\phi}_\theta(\rho,\zeta,\theta)] = 0$$

(20)

The solutions of Eq. (19) will be studied by means of a two sided Laplace transform for the ρ direction. For this purpose we set

$$\rho = \exp t \tag{21}$$

and

$$\eta = \exp \tau \tag{22}$$

Then, by Eq. (17)

$$t = \tau + \log h \tag{23}$$

One obtains from Eq. (15)

$$(f'-n^2\zeta^2)\tilde{\phi}_{\zeta\zeta}(\tau,\zeta,\theta) + [f" + (-n^2-n+\alpha n)\zeta]\,\tilde{\phi}_{\zeta}(\tau,\zeta,\theta)$$

$$+ 2n\zeta\tilde{\phi}_{\zeta,\tau}(\tau,\zeta,\theta) + (-\alpha+1)\tilde{\phi}_{\tau}(\tau,\zeta,\theta)$$

$$- \tilde{\phi}_{\tau\tau}(\tau,\zeta,\theta) - \alpha\,\tilde{\phi}_{\theta\theta}(\tau,\zeta,\theta) = 0 \tag{24}$$

and from Eq. (20)

$$\frac{\partial}{\partial\zeta}[\exp(\beta t)\ \frac{f'-n^2\zeta^2}{h^{n+\alpha-1}}\ \bar{\phi}_{\zeta}(t,\zeta,\theta)] - \frac{\partial}{\partial t}[\exp(\beta t)\frac{f'}{(f'-n^2\zeta^2)h^{n+\alpha-1}}\ \bar{\phi}_{t}(t,\zeta,\theta)]$$

$$- \alpha\ \frac{\partial}{\partial\theta}[\exp(\beta t)\ \frac{1}{h^{n+\alpha-1}}\ \bar{\phi}_{\theta}(t,\zeta,\theta)] = 0 \tag{25}$$

with

$$\beta = (n+\alpha-1) \tag{26}$$

Now the specific form of the function f is introduced. The function $\bar{f}$ can be expressed in the following parametric form.

For $\alpha = 0$, $n = 4/5$

$$\bar{\zeta} = -(1/2)\sigma^{-2/5}\ (2-\sigma)$$

$$\bar{f} = (1/24)\sigma^{-1/5}\ (\sigma^2-6\sigma+48) \tag{27}$$

$$d\bar{\zeta}/d\sigma = (3/10)\ \sigma^{-7/5}(\sigma+(4/3))$$

The following expression (for α=1) is due to Randall.

For $\alpha = 1$ $n = 4/7$

$$\bar{\zeta} = -2\sigma^{-2/7}\ ((1/2)-\sigma)$$

$$\bar{f} = -(8/9)\sigma^{1/7}\ (-2\sigma^2+3\sigma+6) \tag{28}$$

$$\frac{d\bar{\zeta}}{d\sigma} = (10/7)\sigma^{-9/7}\ (\sigma+(1/5))$$

We list a number of expression in terms of the variable σ, including some, which will occur later. Let the value of σ for the limiting characteristic be denoted by σ_L.

For $\alpha = 0$

$$\begin{aligned} \sigma_L &= 16/3 \\ h &= \sigma^{-1/2}(\sigma_L-\sigma)^{-5/6} \\ d\bar{f}/d\bar{\zeta} &= (1/4)\sigma^{1/5}(\sigma-4) \\ d\bar{f}/d\bar{\zeta} - n^2\bar{\zeta}^2 &= -(9/100)\sigma^{-4/5}(\sigma+(4/3)\ (\sigma_L-\sigma) \end{aligned} \tag{29}$$

$$\beta = -(1/5)$$

$$a(\sigma) = \sigma^{1/2}(\sigma_L-\sigma)^{5/6} \tag{30}$$

$$b(\sigma) = (25/9)\sigma^{-1/2}(4-\sigma)(\sigma_L-\sigma)^{-7/6}$$

$$c(\sigma) = 0$$

For $\alpha = 1$

$$\sigma_L = 6/5$$

$$h = \sigma^{-1/2}(\sigma_L - \sigma)^{-7/10} \tag{31}$$

$$d\overline{f}/d\overline{\zeta} = -(8/3)\ \sigma^{3/7}(-\sigma+1)$$

$$d\overline{f}/d\overline{\zeta} - n^2\overline{\zeta}^2 = -(200/147)\sigma^{-4/7}(\sigma+(1/5))(\sigma_L-\sigma)$$

$$\beta = 4/7$$

$$a(\sigma) = \sigma(\sigma_L-\sigma)^{7/5}$$

$$b(\sigma) = (147/50)(1-\sigma)(\sigma_L-\sigma)^{-3/5} \tag{32}$$

$$c(\sigma) = (3/2)\sigma^{-1}(\sigma+\tfrac{1}{5})(\sigma_L-\sigma)^{2/5}$$

The point $\zeta = -\infty$ corresponds to $\sigma = 0$. There one has for $\alpha = 0$

$$\zeta = (\gamma+1)^{-1/3}\ x\ y^{-4/5} = -\mu\ \sigma^{-2/5} + \ \ldots$$

Hence

$$\sigma = (\gamma+1)^{5/6}\ (-x/\mu)^{-5/2}\ y^2 + \ldots$$

and for $\alpha = 1$ (33)

$$\sigma = (\gamma+1)^{7/6}(-x/\mu)^{-7/2}\ y^2 + \ldots$$

We observed above that for the limiting characteristic $(n^2\zeta^2-f') = 0$. One recognizes from the last of the Eqs. (29) and (30), that this is indeed the case for $\sigma = \sigma_L$.

One uses the above formula to replace the independent variable ζ by σ. The primary reason is, of course, the fact that one then obtains a simple representation for f. The introduction of σ (more clearly of $\sigma^{1/2}$) has the further advantage that according to Eqs. (33) the lines $\sigma^{1/2}$ = const sweep out the upper half of xy plane from the negative x axis to the limiting characteristic in a smooth manner as σ changes from 0 to σ_L. In contrast, ζ tends to negative infinity as one approaches the negative x axis. In this regard $\sigma^{1/2}$ would be a preferable independent variable, we retain σ because of certain other advantages.

One has according to the above formulae

for the negative axis	$\sigma = 0$	
for the η axis	$\sigma = 2;$	$\alpha = 0$
	$\sigma = 1/2;$	$\alpha = 1$
for the sonic line ($f' = 0$)	$\sigma = 4;$	$\alpha = 0$
	$\sigma = 1;$	$\alpha = 1$
for the limiting characteristic	$\sigma = 1;$	$\alpha = 1$
	$\sigma = \sigma_L = 16/3;$	$\alpha = 0$
	$\sigma = \sigma_L = 6/5;$	$\alpha = 1$

The values of σ would change if one defines the function $\bar{f}(\bar{\zeta})$ differently (by replacing μ by μ times a constant). Certain simplifications would result if one would make $\sigma_L = 1$.

The coordinate changes described above amount to a mapping from the xy plane to the $\rho\sigma$ plane. One readily derives the following relations:

for $\alpha = 0$

$$y = \rho\sigma^{1/2}(\sigma_L - \sigma)^{5/6}$$

$$x = -(\gamma+1)^{1/3}\mu\ \rho^{4/5}(1 - \sigma/2)(\sigma_L - \sigma)^{2/3}$$

for $\alpha = 1$

$$\eta = \rho\ \sigma^{1/2}(\sigma_L - \sigma)^{7/10} \tag{34}$$

$$x = -(\gamma+1)^{1/3}\ \mu\ \rho^{4/7}(1 - 2\sigma)(\sigma_L - \sigma)^{2/5}$$

The inverse transformation (needed in the application) is more complicated. Assume that μ is known, then from Eqs. (7) and (10)

$$\bar{\zeta} = \mu^{-1}(\gamma + 1)^{-1/3} x\ \eta^{-n} \tag{35}$$

The values of σ is then obtained from the expressions for $\bar{\zeta}$ given in Eqs. (27) and (28). With these values one obtains from Eqs. (17), (29) and (31)

for $\alpha = 0$

$$\rho = y\ \sigma^{-1/2}\ (\sigma_L - \sigma)^{-5/6}$$

for $\alpha = 1$ (36)

$$\rho = r\ \sigma^{-1/2}\ (\sigma_L - \sigma)^{-7/10}$$

The independent variable t is expressed in terms of ρ by Eq. (21).

Notice that for bounded ρ both x and y tend to zero as σ approaches the value σ_L. At the sonic line one finds for ρ = const that $dx/d\sigma = 0$; this means that the line ρ = const has a vertical tangent.

Replacing the independent variable ζ by σ one obtains from Eq. (25)

$$\frac{\partial}{\partial\sigma}[\exp(\beta t)\ a(\sigma)\phi_\sigma(t,\sigma,\theta)] + \frac{\partial}{\partial t}[\exp(\beta t)\ b(\sigma)\phi_t(t,\sigma,\theta)]$$

$$+ \frac{\partial}{\partial\theta}[\exp(\beta t)\ c(\sigma)\phi_\theta(t,\sigma,\theta)] = 0 \qquad (37)$$

The definitions for β, $a(\sigma)$, $b(\sigma)$ and $c(\sigma)$ are found in Eqs. (30) and (32).

The relations to be derived now, are helpful in expressing the farfield conditions, which appear originally in the t,σ,θ system in terms of physical coordinates. Moreover they provide for some flexibility desirable for practical applications. Let L be the differential operator acting in Eq. (12) on ϕ. Consider a fixed curve C in the x, y, z space and two surfaces S_1 and S_2

which have C as boundary. It is assumed that the surface S_1 and its normal vector can be brought into the surface S_2 and its normal vector by a continuous deformation. Let R be the volume swept out by this deformation. Assume that within R

$$L(\phi) = 0 \text{ and } L(\omega) = 0 \tag{38}$$

where ω is some function which satisfies this differential equation. Then one obtains from

$$\iiint_R L(\phi)\omega \, dxdydz = 0$$

by carrying out integrations by part

$$I_{1,S_1} = I_{1,S_2} \tag{39}$$

where

$$I_{1,S_1} = \iint_{S_1} [-(\gamma+1)\Phi_{o,x}(\phi_x\omega - \phi\omega_x)\vec{e}_x + (\phi_y\omega - \phi\omega_y)\vec{e}_y + (\phi_z\omega - \phi\omega_z)\vec{e}_z] \cdot d\vec{o} \tag{40}$$

and I_{1,S_2} is the corresponding integral for S_2. Here, $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ are unit vectors in the respective coordinate directions and $d\vec{o}$ is the directed surface element (pertaining either to S_1 or to S_2). According to Eq. (39) the integral $I_{1,S}$ is invariant against a deformation of S, provided that the boundary C remains unchanged.

For plane flows the surfaces S are replaced by curves connecting two fixed points. Let one such curve be represented by $x=x(p)$, $y=y(p)$ where p is a suitable monotonically changing parameter. Let the values of p pertaining to the end points of this curve be given by p_1 and p_2. Then one has

$$I_1 = \int_{p_1}^{p_2} [-(\gamma+1)\Phi_{o,x}(\phi_x\omega - \phi\omega_x)\vec{e}_x + (\phi_y\omega - \phi\omega_y)\vec{e}_y]\cdot\vec{n}dp$$

$$= \int_{p_1}^{p_2} [-(\gamma+1)\Phi_{o,x}(\phi_x\omega - \phi\omega_x)(dy/\partial p) - (\phi_y\omega - \phi\omega_y)(dx/\partial p)]\partial p \tag{41}$$

In writing down the corresponding relations in cylindrical coordinates for the three dimensional case, it suffices if one uses axisymmetric surfaces S_1 and S_2. The vector normal to these surfaces is then perpendicular to the unit vector in the θ direction. The potential for the basic flow, Φ_o, is axisymmetric, but the perturbations need not be axisymmetric. Let the surfaces S_1 or S_2 be given by $x = x(p)$, $r = r(p)$. One then obtains from Eq. (12)

$$I_2 = \int_{\theta=0}^{\theta=2\pi} \int_{p_1}^{p_2} -(\gamma+1)r\ \Phi_{o,x}(\phi_x\omega - \phi\omega_x)drd\theta - r(\phi_y\omega - \phi\omega_y)dxd\theta \tag{42}$$

The expression, Eq. (42), can, of course, be obtained from Eq. (40) by specialization to an axisymmetric surface.

The corresponding relation obtained from Eq. (25), again for an axisymmetric surface, is:

$$I_3 = \int_{\theta=0}^{2\pi} \int_{p_1}^{p_2} \exp(\beta t)\left[\frac{f'-n^2\zeta^2}{h^{n+\alpha-1}}(\phi_\zeta\omega - \phi\omega_\zeta)dtd\theta + \frac{f'}{(f'-n^2\zeta^2)h^{n+\alpha-1}}(\phi_t\omega - \phi\omega_t)d\zeta d\theta\right] \tag{43}$$

where ζ and t are considered as functions of the parameter p. Here $\phi = \phi(t,\zeta,\theta)$ and $\omega = \omega(t,\zeta,\theta)$ and these functions satisfy Eq. (25). For two dimensional flow ($\alpha = 0$), the dependence upon the third coordinate (here θ) does not appear and the integration over θ is omitted.

Finally, one has as a corresponding expression in the t,σ,θ system

$$I_4 = \int_{\theta=\theta}^{\theta=2} \int_{p_1}^{p_2} \exp(\beta t)[a(\sigma)(\phi_\sigma \omega - \phi\omega_\sigma)dtd\theta - b(\sigma)(\phi_t\omega - \phi\omega_t)d\sigma d\theta] \tag{44}$$

Eqs. (41) through (44) are, of course, in essence the same except for the choice of the coordinates. All arise from a volume integral of the divergence of the same vector multiplied by ω. They may, however, differ by constant factors because such factors have been disregarded in going from one form of the differential equation to another one. One has

$$I_3 = -(\gamma+1)^{-1/3} I_1$$

$$I_3 = -\mu(3/10) I_4 \quad \text{for } \alpha = 0 \tag{45}$$

$$I_3 = -\mu(20/21) I_4 \quad \text{for } \alpha = 1$$

We shall find that the formulation of far field conditions leads to expressions of the form (44). The surface S represents the distant boundary of the computed flow field, ϕ is the perturbation potential, and ω is given by certain particular solutions of the equation for the perturbation potential which plays the role of test functions. The theory will be developed in the σ,t system, while for the practical applications the xyz system will be used. Equation (39) provides flexibility in the choice of the distant boundary surface.

SECTION III

FORMULATION OF A BOUNDARY VALUE PROBLEM

The basis for the future discussions is Eq. (37). This is a partial differential equation in the σ, t, θ space. In principle, one might also use Eq. (25), but the introduction of σ instead of ζ simplifies the evaluation. The field between the negative x axis and the limiting characteristic is mapped into the strip $0 \leq \sigma \leq \sigma_L$. In the following derivations we assume that the distant limit of the computed flow field is axisymmetric, although the flow itself need not be axisymmetric. The map of the outer limit of the computed flow field is then a curve C in the σ, t plane which starts at the negative x axis ($\sigma = 0$), and ends at the limiting characteristic. Let this curve be given by

$$t = t_S(\sigma).$$

(For a field without axial symmetry the outer surface of the computed flow field maps into a cylindrical surface to be denoted by S in the t, σ, θ space with the curve C as a directrix, and lines parallel to the θ axis as generators. In this case, one has $0 \leq \theta \leq 2\pi$, with periodicity conditions at $\theta = 0$, and $\theta = 2\pi$.) The far field conditions will appear as global relations between ϕ and its gradient at this surface. At the boundary $\sigma = 0$, one has for $\alpha = 0$ (plane flows) the requirement that the solution be either symmetric or antisymmetric (actually with respect to $\sigma^{1/2}$), for the case $\alpha = 1$ (three dimensional flows), one has the requirement that the solution be free of singularities. Details will appear in the course of the investigation. Boundary conditions at the limiting characteristic will be formulated later.

The far field maps into the region to the right of the surface S, $(t = t_S(\sigma))$. The function ϕ is continued to the left by setting $\phi = 0$.

The function ϕ so defined satisfies the differential equation (37) in the whole strip $0 \leq \sigma \leq \sigma_L$ except for the surface S where ϕ and gradient ϕ have jumps. Let

$$\Delta\phi(\sigma,\theta) = \phi(t_s(\sigma) + 0,\sigma, \theta) - \phi(t_s(\sigma) -0, \sigma,\theta)$$

$$\Delta\phi_t(\sigma,\theta)=\phi_t(t_s(\sigma)+0,\sigma,\theta) - \phi_t(t_s(\sigma)-0,\sigma,\theta) \tag{46}$$

$$\Delta\phi_\sigma(\sigma,\theta)=\phi_\sigma(t_s(\sigma) + 0, \sigma, \theta) -\phi_\sigma(t_s(\sigma) - 0, \sigma,\theta)$$

There exists, of course, a relation between $\Delta\phi_t$, $\Delta\phi_\sigma$, and $\Delta\phi$) The θ component of the gradient does not enter, because the surface S is assumed to be axisymmetric.

For large values of t the perturbations to the basic flow are allowed to have at most the same order of magnitude as the basic flow itself. The form of the basic flow is given by Eq. (6). One has $n = 4/5$ for $\alpha = 0$ and $n = 4/7$ for $\alpha= 1$. η is connected with t by Eqs. (17) and (21). Along a line σ = constant (which coincides with a line ζ = constant), the potential of the basic flow therefore behaves as exp (2t/5) for $\alpha = 0$, and exp (-2t/7) for $\alpha = 1$. We therefore impose the conditions

$$\exp(-2t/s)\phi(t,\sigma,\theta) \text{ bounded for } t\to +\infty;\alpha = 0$$

$$\exp(2t/7)\phi(t,\sigma,\theta) \text{ bounded for } t\to + \infty;\alpha = 1. \tag{47}$$

For $t < t_s(\sigma)$ one has $\phi= 0$, by definition. This is compatible with the postulate

$$\exp((-2/5)-\varepsilon)t)\phi(t,\sigma,\theta) \to 0 \text{ for } t\to - \infty;\ \alpha= 1$$

$$\exp((2/7)-\varepsilon)t)\phi(t,\sigma,\theta) \to 0 \text{ for } t\to - \infty;\ \alpha=1,\ \varepsilon > 0. \tag{48}$$

The problem defined by Eqs. (45), (47), and (48) is always solvable even for arbitrary functions $\Delta\phi$, $\Delta\phi_t$, and $\Delta\phi_\sigma$, but then there is no guarantee that $\phi \equiv 0$ for $t < t_s(\sigma)$.

In order to examine whether a set of functions $\phi(t_s(\sigma), \sigma,\theta)$ $\phi_t(t_s(\sigma)\sigma,\theta)$ and $\phi_\sigma(t_s(\sigma),\sigma,\theta)$ pertain to a field which satisfies the far field conditions, we set $\Delta\phi$, $\Delta\phi_t$, and $\Delta\phi_\sigma$ equal to these functions ϕ, ϕ_t, and ϕ_σ and solve the above problem. The function $\phi(t, \phi_t, \theta)$ so obtained will satisfy Eq. (47); these are the far field conditions; but only if $\phi \equiv 0$ to the left of the surface $t = t_s(\sigma)$ will the limiting values of ϕ, ϕ_t, and ϕ_σ obtained as one approaches this surface from the right, be equal to the functions $\phi(t_s, \sigma,\theta)$, $\phi_t(t_s,\sigma,\theta)$ and $\phi_\sigma(t_s,\sigma,\theta)$ which are to be examined, and with which the computation started.

Conditions on $\phi(t_s,\sigma,\theta)$, $\phi_t(t_s\sigma,\theta)$ and $\phi_\sigma(t_s,\sigma,\theta)$ which ensure that $\phi \equiv 0$ to the left of $t = t_s(\sigma)$ will therefore constitute the desired far field conditions. In the next section such conditions will be derived by means of Laplace transform.

In the case $\alpha = 0$, the sonic line (of the basic flow) is given by $\sigma = 4$. This can also be seen from Eq. (37); the coefficients $a(\sigma)$ and $b(\sigma)$ of $\phi_{\sigma\sigma}$ and ϕ_{tt} have according to Eq. (30), the same sign for $\sigma < 4$ (elliptic region) and the opposite sign for $\sigma > 4$ (hyperbolic region). The direction of the characteristics is determined by the specific form of the functions $a(\sigma)$ and $b(\sigma)$. One obtains

$$\frac{dt}{d\sigma} = \pm \frac{5}{3} \frac{(\sigma-4)^{1/2}}{\sigma^{1/2}(16/3)-\sigma)}, \quad \alpha = 0 \tag{49}$$

Hence, for the vicinity of $\sigma = \sigma_L = 16/3$

$$t = \pm \frac{5}{6}\log (\sigma_L - \sigma) + \text{const.}$$

As $\sigma \to \sigma_L$, t approaches $\pm \infty$ for the two families of characteristics. Incidentally, the integration required in Eq. (49) can be carried out in closed form

$$t = \pm\{(10/3)\log[\sigma^{1/2} - (\sigma-4)^{1/2}] + (5/3 \log [\sigma^{1/2} + 2(\sigma-4)^{1/2}] - 5/6 \log[16/3) - \sigma]\} + \text{const.} \tag{50}$$

We mentioned before that because of Eq. (21) for each finite ρ, which is equivalent to finite t, $x \to 0$, $y \to 0$, or $r \to 0$, as $\sigma \to \sigma_L$. Obviously, the outer boundary of the computed flow field given by the surface S cannot extend to the origin as one approaches the limiting characteristic. It will terminate there at a finite value of y. This behavior is more clearly represented in a σ,τ plane (τ is defined in Eq. (22). Instead of Eq. (37), which originates from Eq. (25) one would then deal with the counter part of Eq. (24). One has for the characteristics (from Eqs. (23) and (49)

$$d\tau/d\sigma = dt/d\sigma - d\log h/d\sigma = (1/3)\sigma^{-1}(16/3-\sigma)^{-1} \left[\pm 5\sigma^{1/2}(\sigma-4)^{1/2} + 4(2-\sigma)\right] \tag{51}$$

The term in the bracket is regular for $\sigma = 16/3$; it has a zero of the first order if one chooses the positive sign. This cancels the factor $((16/3)-\sigma)^{-1}$ in front of the bracket. For this family of characteristics τ assumes a finite value at the limiting characteristic.

For $\alpha = 1$, one first carries out a Fourier decomposition with respect to the θ direction and considers each component separately. One then obtains analogous results.

SECTION IV
LAPLACE TRANSFORMATION

The problem formulated in Section III is now treated by a two sided Laplace transform. The starting point is Eq. (37). First we make a Fourier decomposition of $\phi(t,\sigma,\theta)$ with respect to θ.

$$\phi(t,\sigma,\theta) = \sum_{m=-\infty}^{\infty} \psi^{(m)}(t,\sigma) \exp(im\theta) \tag{52}$$

Hence

$$\phi^{(m)}(t,\sigma) = \frac{1}{2\pi} \int_0^{2\pi} \phi(t,\sigma,\theta) \exp(-im\theta) d\theta \tag{53}$$

The Laplace transform of $\phi^{(m)}(t,\sigma)$ is defined by

$$\psi^{(m)}(s,\sigma) = \int_{-\infty}^{\infty} \phi^{(m)}(t,\sigma) \exp(-st) dt \tag{54}$$

We substitute Eq. (52) into Eq. (37) and consider only one Fourier component. The resulting equation is multiplied by $\exp(-(s+\beta)t)$ and integrated from $t = -\infty$ to $t = +\infty$. One then obtains

$$\begin{aligned} \int_{-\infty}^{t_s(\sigma)} + \int_{t_s(\sigma)}^{+\infty} & \left\{ \frac{\partial}{\partial\sigma} (a(\sigma)\phi_\sigma^{(m)}) + b(\sigma)(\beta\phi_t^{(m)} + \phi_{tt}^{(m)}) \right. \\ & \left. -m^2 c(\sigma)\phi^{(m)} \right\} \exp(-st)dt = 0 \end{aligned} \tag{55}$$

In rewriting this equation in terms of $\psi^{(m)}$ one must take into account that $\phi^{(m)}$, $\phi_t^{(m)}$ and $\phi_\sigma^{(m)}$ are discontinuous along the curve $t = t_s(\sigma)$. The individual terms are considered separately. One has the trivial result

$$\int_{-\infty}^{t_s(\sigma)} + \int_{t_s(\sigma)}^{+\infty} (-m^2 c(\sigma)\phi^{(m)}(t,\sigma)\ \exp(-st)dt = m^2 c(\sigma)\psi^{(m)}(s,\sigma)) \quad (56)$$

A familiar procedure gives

$$\int_{-\infty}^{t_s(\sigma)} + \int_{t_s(\sigma)}^{+\infty} b(\sigma)(\beta\phi_t^{(m)}(t,\sigma) + \phi_{tt}^{(m)}(t,\sigma))\ \exp(-st)dt$$

$$= b(\sigma)\Big\{-[(\beta+s)\Delta\phi^{(m)}(\sigma) + \Delta\phi_t^{(m)}(\sigma)]\ \exp(-st_s(\sigma))$$

$$+ (\beta s + s^2)\psi^{(m)}(s,\sigma)\Big\} \quad (57)$$

The notation $\Delta\phi^{(m)}$ etc. used here is analogous to the definition Eq. (46). For the evaluation of the remaining term we rewrite the following expression in terms of $\phi^{(m)}(t,\sigma)$

$$\frac{\partial}{\partial\sigma}[a(\sigma)\frac{\partial\psi^{(m)}(s,\sigma)}{\partial\sigma}] = \frac{\partial}{\partial\sigma}\Big\{a(\sigma)\ \frac{\partial}{\partial\sigma}\Big[\int_{-\infty}^{t_s(\sigma)} + \int_{t_s(\sigma)}^{+\infty} \phi^{(m)}(t,\sigma)\exp(-st)dt\Big]\Big\}$$

$$= \frac{\partial}{\partial\sigma}\Big\{-a(\sigma)\frac{\partial t_s}{\partial\sigma}\ \Delta\phi^{(m)}(\sigma)\exp(-st_s(\sigma) + a(\sigma)\Big[\int_{-\infty}^{t_s(\sigma)} + \int_{t_s(\sigma)}^{+\infty} \phi_\sigma^{(m)}(t,\sigma)\exp(-st)dt\Big]\Big\}$$

$$= \frac{\partial}{\partial\sigma}\{-a(\sigma)(dt_s/d\sigma)\Delta\phi^{(m)}(\sigma)\exp(-st_s(\sigma)\} - a(\sigma)(dt_s/d\sigma)\Delta\phi_\sigma^{(m)}(\sigma)\exp(-st_s(\sigma)$$

$$+ a'(\sigma)\Big[\int_{-\infty}^{t_s(\sigma)} + \int_{t_s(\sigma)}^{+\infty} \phi_\sigma^{(m)}(t,\sigma)\exp(-st)dt\Big] + a(\sigma)\Big[\int_{-\infty}^{t_s(\sigma)} \int_{t_s(\sigma)}^{+\infty} \phi_{\sigma\sigma}^{(m)}\ (t,\sigma)\exp(-st)dt\Big]$$

$$= \frac{\partial}{\partial\sigma}\{-a(\sigma)(dt_s/d\sigma)\Delta\phi^{(m)}(\sigma)\exp(-st_s(\sigma)\} - a(\sigma)(dt_s/d\sigma)\Delta\phi_\sigma^{(m)}(\sigma)\exp(-st_s(\sigma)$$

$$+ \int_{-\infty}^{t_s(\sigma)} \int_{t_s(\sigma)}^{+\infty} \frac{\partial}{\partial\sigma}[a(\sigma)\ \phi_\sigma^{(m)}(t,\sigma)]\exp(-st)dt \quad (58)$$

Substituting Eqs. (56) through (58) into Eq. (55), one obtains

$$\frac{\partial}{\partial\sigma}\,[a(\sigma)\psi_{\sigma}^{(m)}(s,\sigma)] + [b(\sigma)(\beta s + s^2) - m^2 c(\sigma)]\psi^{(m)}(s,\sigma)$$

$$= r^{(m)}(s,\sigma) \tag{59}$$

with

$$r^{(m)}(s,\sigma) = b(\sigma)[(\beta+s)\Delta\phi^{(m)}(\sigma) + \Delta\phi_{t}^{(m)}(\sigma)]\exp(-st_s(\sigma))$$

$$+ \frac{\partial}{\partial\sigma}\{-a(\sigma)(dt_s/d\sigma)\Delta\phi^{(m)}(\sigma)\exp(-st_s(\sigma)\}-a(\sigma)(dt_s/d\sigma)\Delta\phi_{\sigma}^{(m)}(\sigma)\exp(-st_s(\sigma) \tag{60}$$

Eq. (59) is an ordinary differential equation for $\psi^{(m)}$. The argument s plays the role of a parameter. The boundary condition for $\sigma = 0$ arises from the properties of the function $\phi^{(m)}(t,\sigma)$. For $\alpha = 0$ one obtains the requirement that ψ, considered as a function of $\sigma^{1/2}$, be either symmetric or antisymmetric with respect to $\sigma = 0$. For $\alpha = 1$ one has the requirement that ψ be nonsingular.

To obtain conditions at the limiting characteristic, we consider the problem in the τ,σ plane. One has, of course,

$$\phi^{(m)}(t,\sigma) = \tilde{\phi}^{(m)}(\tau,\sigma)$$

We define

$$\tilde{\psi}^{(m)}(s,\sigma) = \int_{-\infty}^{+\infty} \tilde{\phi}^{(m)}(\tau,\sigma)\ \exp(-s\tau)d\tau \tag{61}$$

One finds with Eqs. (17), (21), and (22)

$$\tilde{\psi}^{(m)}(s,\sigma) = h^{s}\ \psi^{(m)}(s,\sigma) \tag{62}$$

According to Eq. (22), τ is finite if y is finite. This holds in particular at the intersection of the surface S with the limiting characteristic. For finite τ the function $\phi^{(m)}$ is bounded along the limiting characteristic. The behavior of ϕ postulated in Eqs. (47) and (48) carries over to $\phi^{(m)}$ and holds also in terms of τ. Hence it follows that $\bar{\psi}^{(m)}$ is bounded

$$\begin{aligned} &\text{for } (2/5) < \mathrm{Re}(s) < (2/5) + \varepsilon, \text{ if } \alpha = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \varepsilon > 0 \\ &\text{and for } (-2/7) < \mathrm{Re}(s) < -(2/7) + \varepsilon, \text{ if } \alpha = 1 \end{aligned} \tag{63}$$

The behavior of $\psi^{(m)}$ is then found from Eq. (62). We shall refer to the strip in the s plane characterized by Eqs. (63) as the original region of definition of the Laplace transform. The Laplace transform in the remainder of the s plane is obtained by analytic continuation.

SECTION V

THE HOMOGENEOUS PART OF EQ (59)

In preparation of the evaluation of $\psi^{(m)}(s,\sigma)$, we discuss the solutions for the homogeneous part of Eq. (59). We write this equation in the form

$$\frac{d}{d\sigma}\,(a(\sigma)\,(dG^{(m)}/d\sigma) + [\lambda b(\sigma) - m^2 c(\sigma)]G^{(m)} = 0 \tag{64}$$

with

$$\lambda = s(s + \beta) \tag{65}$$

Important for applications is that, because of the special form of $a(\sigma)$, $b(\sigma)$ and $c(\sigma)$, defined in Eqs. (30) and (32) this is a hypergeometric differential equation. If one sets in analogy to Eq. (62)

$$g^{(m)}(\sigma) = h^s\, G^{(m)}(\sigma)$$

$$= \begin{cases} \sigma^{-s/2}(\sigma_L-\sigma)^{-5s/6}\, G(\sigma); & \alpha = 0 \\ \sigma^{-s/2}(\sigma_L-\sigma)^{-7s/10}\, G^{(m)}(\sigma); & \alpha = 1 \end{cases} \tag{66}$$

then one obtains the following differential equations

$$\sigma^2((16/3)-\sigma)(d^2g/d\sigma^2) + ((4/3)+(8/3)s)\sigma(2-\sigma)(dg/d\sigma)$$
$$+ s(s-1)(\sigma+(4/3))g = 0, \qquad \alpha = 0 \tag{67}$$

$$\sigma^2(6-5\sigma)(d^2g^{(m)}/d\sigma^2) + 12(s+1)\sigma(1-2\sigma)(dg^{(m)}/d\sigma)$$
$$+ 3(s^2-m^2)(1+5\sigma)g^{(m)} = 0, \qquad \alpha = 1 \tag{68}$$

The second equation is given in the article by Randall.

One readily verifies that

$$\phi = G^{(m)}(\sigma)\exp(st)\exp(im\theta) = g^{(m)}(\sigma)\exp(s\tau)\exp(im\theta)$$
$$= g^{(m)}(\sigma)y^{s}\exp(im\theta) \tag{69}$$

and (because of Eqs. (65) and (23))

$$\phi = G^{(m)}(\sigma)\exp(-(s-\beta)t)\exp(im\theta) = h^{-2\rho-\beta}g^{(m)}(\sigma)y^{s-\beta}\exp(im\theta)$$

are particular solutions of Eq. (37), and of Eq. (13) if one returns to the coordinates of the physical plane. The second form is singular at the limiting characteristic if the first form is regular, because of the factor $h^{-2s-\beta}$. The totality of the solutions of Eqs. (67) and (68) are expressed by Riemann's P-functions,

$$g = P\begin{bmatrix} 0 & \infty & 16/3(=\sigma_L) & \\ -s/2 & -(s/3)+(1/3) & 0 & \sigma \\ -(s/2)+(1/2) & 3s & (1/6)-(5s/3) & \end{bmatrix} \quad \alpha = 0 \tag{70}$$

$$g^{(m)} = P\begin{bmatrix} 0 & \infty & 6/5(=\sigma_L) & \\ -(s/2)+(m/2) & (6s/5)+(7/10)+R & 0 & \sigma \\ -(s/2)-(m/2) & (6s/5)+(7/10)-R & -(2/5)-(7s/5) & \end{bmatrix} \quad \alpha = 1 \tag{71}$$

where

$$R = [-(3/2)m^2 + (1/4) + (3/50)(7s + 2)^2]^{1/2} \tag{72}$$

Because of the form of h (Eqs. (20) and (31)), the differential function G satisfies a hypergeometric equation with the same singular points but different exponents. For the vicinity of $\sigma = 0$, the exponents of the differential equation for G are increased by s/2; particular solutions for G, therefore have the form $G = P(\sigma)$ and $G = \sigma^{1/2}P(\sigma)$ for $\alpha = 0$, and $P = \sigma^{m/2}P(\sigma)$ and $P = \sigma^{-m/2}P(\sigma)$ for $\alpha = 1$. The solutions for $\alpha = 1$ which are free of singularities start with the term $\sigma^{m/2}$. Power series which start with the constant term are denoted by $P(\sigma)$, used in a generic sense.

We use the self-explanatory notation

$$\left.\begin{aligned} g &= \sigma^{-s/2}\, \hat{g}_{sym} \\ g &= \sigma^{-(s/2)+(1/2)}\, \hat{g}_{anti} \end{aligned}\right\} \quad \alpha = 0 \tag{73}$$

$$g^{(m)} = \sigma^{-(s/2)+(m/2)}\, \hat{g}^{(m)} \qquad \alpha = 1$$

$$\left.\begin{aligned} G_{sym} &= (\sigma_L-\sigma)^{5s/6}\, \hat{g}_{sym} \\ G_{anti} &= (\sigma_L-\sigma)^{5s/6}\sigma^{1/2}\, \hat{g}_{anti} \end{aligned}\right\} \quad \alpha = 0 \tag{74}$$

$$G^{(m)} = (\sigma_L-\sigma)^{7s/10}\, \sigma^{m/2}\, \hat{g}^{(m)} \qquad \alpha = 1$$

Then one has for the totality of the respective solutions

$$\hat{g}_{sym} = P\begin{bmatrix} 0 & \infty & 1 & \\ 0 & -(5s/6)+(1/3) & 0 & \sigma/\sigma_L \\ 1/2 & 5s/2 & (1/6)-(5s/3) & \end{bmatrix}$$

$$\hat{g}_{anti} = P\begin{bmatrix} 0 & \infty & 1 & \\ 0 & -(5s/6)+(5/6) & 0 & \sigma/\sigma_L \\ -1/2 & (5s/2)+(1/2) & (1/6)-(5s/3) & \end{bmatrix}$$

$$\hat{g}^{(m)} = P\begin{bmatrix} 0 & \infty & 1 & \\ 0 & (m/2)+(7s/10)+(7/10)+R & 0 & \sigma/\sigma_L \\ -m & (m/2)+(7s/10)+(7/10)-R & -(2/5)-(7s/5) & \end{bmatrix}$$

Now, the solution with exponent zero at $z = 0$ for an expression

$$P\begin{bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{bmatrix}$$

is given by Gauss' hypergeometric series

$$F(a,b,c,z) = 1 + \frac{ab}{c1!} z + \frac{a(a+1)b(b+1)}{c(c+1)2!} z^2 + \ldots \qquad (75)$$

Of course, the notation a,b,c for the parameters of the hypergeometric series (which is traditional) has nothing to do with the functions $a(\sigma)$, $b(\sigma)$, $c(\sigma)$ defined in Eqs. (30) and 32).

Then one obtains

$$\begin{aligned} &\text{for } \hat{g}_{sym}: a = -(5s/6) + (1/3);\ b = (5s/2);\ c = 1/2 \\ &\text{for } \hat{g}_{anti}: a = -(5s/6) + (5/6);\ b = (5s/2) + (1/2);\ c = 3/2 \end{aligned} \qquad (76)$$

$$\text{for } \hat{g}^{(m)} \quad \begin{aligned} a &= (m/2) + (7s/10) + (7/10) + R \\ b &= (m/2) + (7s/10) + (7/10) - R \\ c &= m + 1 \end{aligned}$$

For $\sigma = \sigma_L$ one of the exponents is zero, the other exponent is given by

$$c - a - b = (1/6) - (5s/3) \qquad \text{for } \alpha = 0$$

and

$$c - a - b = (-2/5) - (7s/5) \qquad \text{for } \alpha = 1$$

One has the following relations between the hypergeometric series at $z = 0$ and $z = 1$

$$F(a,b,c,z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b, 1 + a + b - c, 1 - z)$$

$$+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} F(c-a, c-b, 1+c-a-b, 1-z)$$

To apply this relation to the function $\mathcal{G}$ we define

$$\begin{aligned} \mathcal{G}_1 &= F(a, b, c, \sigma/\sigma_L) \\ \mathcal{G}_2 &= F(a, b, 1 + a+b-c, (1 - (\sigma/\sigma_L)) \\ \mathcal{G}_3 &= (1-(\sigma/\sigma_L)^{c-a-b} F(c-a, c-b, 1+c-a-b, (1-(\sigma/\sigma_L)) \end{aligned} \tag{77}$$

Then one has

$$\mathcal{G}_1 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \mathcal{G}_2 + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \mathcal{G}_3 \tag{78}$$

We notice that the function $\mathcal{G}_2$ which is regular at $\sigma = \sigma_L$ is dominated in this vicinity by $\mathcal{G}_3$ if $c-a-b < 0$. This happens for Re $s > 1/10$ if $\alpha = 0$ and for Re $s > -2/7$ for $\alpha = 1$. These regions include the original regions of definition of the Laplace transform given in Eq. (63). The same holds for the corresponding functions G (see Eq. (74)).

In Eqs. (69) we had used the functions G to define particular solutions of the differential equation for ϕ. These expressions can be used to obtain some insight into the representation of the flow field. One expects that the function ϕ will have the correct symmetry properties at the negative x axis ($\sigma = 0$) and that it will be free of singularities at the limiting characteristic. For y finite, the first of Eqs. (69) then implies that at the limiting characteristic $g^{(m)}$ must be regular and according to Eqs. (73), this implies that $\hat{g}^{(m)}$ and, of course, also $\hat{g}_{sym}$ and $\hat{g}_{anti}$ must be regular. Such regular solutions are obtained if the series for $\hat{g}$ terminates after a finite number of terms, or that the coefficient of $\hat{g}_3$ in Eq. (78) vanishes. This leads to the condition that either a or b is a negative integer, (provided that (c-a-b) is not a non-negative integer). For $\alpha = 0$, and $\hat{g}_{sym}$, one obtains by Eq. (76) from

$$\begin{aligned} a &= -(5s/6) + 1/3 = -k \\ s &= (6k/5 + (2/5); \; k = 0, 1, 2... \end{aligned} \tag{79}$$

and from

$$\begin{aligned} b &= 5s/2 = -k; \\ s &= -(2k/5), \; k = 0, 1, \;\; 2 \end{aligned} \tag{80}$$

for $\alpha = 0$ and $\hat{g}_{anti}$ one obtains

$$s = (6k/5) + 1; \; k = 0, 1,2...$$

and (81)

$$s = -(2k/5) - (1/5); \; k = 0,1,2...$$

In neither of these cases will c-a-b be an integer. These values of s are evenly spaced. The spacing for negative values of s is 1/3 of that for positive values of s.

One notices that the square root R which occurs in Eq. (71) for $\alpha = 1$, has opposite signs in a and b. Therefore, no square root appears in the coefficients of the hypergeometric series (76).

The condition for the termination of the series for $\hat{g}^{(m)}$, that a or b be a nonpositive integer gives

$$(m/2) + (7s/10) + (7/10) \pm R = -k; \; k = 0,1,2...$$

where R is given by Eq. (72). This is a quadratic equation for s, which ultimately leads to

$$s = (1/7)[(2k+m-1) \pm [24k^2 + 24k(m+1) + (6m+1)^2]^{1/2}, \quad k = 0,1,2 \tag{82}$$

A corresponding formula is found in the article by Randall (set $k = N_o - 1$). One obtains

$$\begin{array}{lll} \text{for} & m = 0,\ k = 0, & s = -2/7 \text{ and } s = 0 \\ \text{for} & m = 0,\ k = 1, & s = -6/7 \text{ and } s = 8/7 \\ \text{for} & m = 0,\ k = 2, & s \text{ irrational} \\ \text{for} & m = 1,\ k = 0, & s = \pm 1 \\ \text{for} & m = 1,\ k = 1, & s = -9/7 \text{ and } s = 13/7 \end{array} \tag{83}$$

Here it can happen that c-a-b is simultaneously a non-negative integral number. Values can be found by a systematic search. One has, for

$$c-a-b = -(7s/5) - (2/5) = k_1 \qquad k_1 = 0,\ 1,\ 2,\ldots$$

$$s = -(5k_1/7) - (2/7)$$

For these values of s one obtains

$$b = (m/2) + (1/2) - (k_1/2) - (1/2)\ [6(k_1^2 - m^2) + 1]^{1/2}$$

For b to be a nonpositive integer it is necessary, but not sufficient, that the radicand $6(k_1^2 - m^2) + 1$ be the square of an integer, say k_3^2. Then

$$(1/6)(k_3^2 - 1) = (1/6)(k_3 - 1)(k_3 + 1)$$

must be an integer. This means that $k_3 + 1$ or $k_3 - 1$ must be divisible by 6.

$$k_3 = 6k_4 \pm 1; \qquad k_4 = 0,\ 1,\ 2$$

Next, one examines whether there are integers k_1 and m so that

$$k_1^2 = (1/6)(k_3^2 - 1) + m^2$$

This is done by systematically varying m. Only a finite number of terms need be examined. If m is increased, then the value of k_1 (obtained from the last equation k_1) increases, but more slowly than that of m. Considering k_1 as a function of m, not necessarily an integer, one has from the last equation

$$dk_1/dm = m[(1/6)(k_3^3 - 1) + m^2]^{-1/2} < 1$$

The search can stop after the difference $k_1 - m$ becomes smaller than 1.

We list some of the values

$k_4 = 0,\ k_3 = \pm 1,\quad m = 0,\ k_1 = 0\quad b = 0$

$\quad m = 1,\ k_1 = 1\quad b = 0$

$k_4 = 1,\ k_3 = 5,\ (1/6)(k_3^2-1) = 4,\ m = 0,\ k_1 = 2,\ b = -3$

$k_3 = 7,\ (1/6)(k_3^2-1) = 8,\ m = 1,\ k_1 = 3,\ b = -4$

$k_4 = 2,\ k_3 = 11,\ (1/6)(k_3^2-1) = 20,\ m = 4,\ k_1 = 6,\ b = -6$

$k_3 = 13,\ (1/6)(k_3^2-1) = 28,\ m = 6,\ k_1 = 8,\ b = -7$

This shows that cases are possible where $-b < c-a-b$ and where $-b \geq c-a-b$. One has in the basic flow (from Eqs. (6) and (9))

$$\Phi_0 = y^{2/5}\ f(\zeta) \quad \text{for} \quad \alpha = 0$$
$$\Phi_0 = r^{-2/7}\ f(\zeta) \quad \text{for} \quad \alpha = 1 \tag{84}$$

One will postulate that none of the perturbation will dominate the basic flow as $y \to \infty$ or $r \to \infty$. According to Eq. (69), one must, therefore, exclude for $\alpha = 0$ particular solutions with Re $s > 2/5$, and for $\alpha = 1$ particular solutions with Re $s > -2/7$.

The role of the particular solutions for negative values of s is not obvious. There is, for instance, the question whether the particular solutions with the special values of s obtained here are suitable to represent the propagation of a singularity along the limiting characteristic. Such a singularity would arise if it is present in the profile shape. The discussion of the problem by means of the Laplace transform which we have chosen here will answer questions of this kind.

Some of these particular solutions have simple interpretations, which are important for the applications.

The particular solution for $s = 0$ gives $\Phi = \text{const}$, this is certainly an admissible (although uninteresting) perturbation. Particular solutions for $\alpha = 0$ and $s = 1$ give $\phi = y$, this trivial solution is excluded because it alters the boundary conditions at infinity.

An expression (for $\alpha = 0$)

$$\Phi = (y - \Delta y^{2/5}(\mu + \Delta\mu)^3 \bar{f}(\bar{\bar{\zeta}}) \tag{85}$$

with

$$\bar{\bar{\zeta}} = (\mu + \Delta\mu)^{-1}(\gamma + 1)^{-1/3}(x - \Delta x)(y - \Delta y)^{-4/5}$$

is a solution of the original problem Eq. (1) with Φ_{zz} omitted, for it is the expression (10) with μ replaced by $\mu + \Delta\mu$ in a system of coordinates in which the origin is shifted by Δx in the x direction and by Δy in the y direction. Considering $\Delta\mu$, Δx, and Δy as small and setting (as before)

$$\bar{\zeta} = \mu^{-1}\zeta = \mu^{-1}(\gamma + 1)^{-1/3} x\ y^{-4/5}$$

one obtains

$$\begin{aligned}\Phi &= (y - \Delta y)^{2/5}(\mu + \Delta\mu)^3 f(\bar{\bar{\zeta}}) \\ &= y^{2/5}\mu^3 \bar{f}(\bar{\zeta}) + \Delta\mu\ \mu^2\ y^{2/5}[3\bar{f}(\bar{\zeta}) - \bar{\zeta}(df/d\bar{\zeta})]\end{aligned}$$

$$- \Delta x\mu^2(\gamma+1)^{-1/3}y^{-2/5}(d\bar{f}/d\bar{\zeta})$$

$$- \Delta y\mu^3\ y^{-3/5}[(2/5)\bar{f} - (4/5)\bar{\zeta}(d\bar{f}/d\bar{\zeta})]$$

$$+ \text{higher order terms}$$

It follows that perturbation solutions with $s = 2/5$ express a change of μ (the intensity of the basic perturbations), symmetric particular solutions with $s = -2/5$ and antisymmetric particular solutions with $s = -3/5$ represent, respectively, a shift of the origin in the x and y directions. Using the Eqs. (27) and (29), one can express these functions in terms of σ and identify them with the functions $\hat{g}_{symm}$ and $\hat{g}_{anti}$ defined above. One obtains, for instance, using the definitions (73)

$$3\bar{f}(\bar{\zeta}) - \bar{\zeta}(d\bar{f}/d\bar{\zeta}) = 5\sigma^{-1/5} = 5^{-1/5}g_{sym}(\sigma,\ 2/5))$$

where the second argument of $\hat{g}$ refers to the value of s. Therefore, ultimately

$$\begin{aligned}\Phi &= (y-\Delta y)^{2/5}(\mu+\Delta\mu)^3\bar{f}(\bar{\bar{\zeta}}) \\ &= y^{2/5}\mu^3\bar{f}(\bar{\zeta}) + \Delta\mu\mu^2\ 5y^{2/5}\sigma^{-1/5}\hat{g}_{sym}(\sigma,\ (2/5)) \\ &+ \Delta x(\gamma+1)^{-1/3}\mu^2\ y^{-2/5}\sigma^{1/5}\hat{g}_{sym}(\sigma,\ -(2/5)) \\ &- \Delta y\ \mu^3(1/2)\ y^{-3/5}\sigma^{(1/2)+(3/10)}\hat{g}_{ant}(\sigma,\ -(3/5)) \\ &+ \ldots\end{aligned} \tag{86}$$

Analogous discussions are carried out for $\alpha = 1$. One starts with

$$\Phi = (\mu + \Delta\mu)^3(\ r + \Delta r)^{-2/7}\bar{f}(\bar{\bar{\zeta}}) \tag{87}$$

with

$$\bar{\bar{\zeta}} = (\mu + \Delta\mu)^{-1}(\gamma + 1)^{-1/3}(x - \Delta x)(r + \Delta r)^{-4/7}$$

and

$$(r + \Delta r)^2 = (y - \Delta y)^2 + (z - \Delta z)^2$$

setting

$$\bar{\zeta} = \mu^{-1}(\gamma + 1)^{-1/3} \, x \, r^{-4/7}$$

Then one obtains by considering $\Delta\mu$, Δx, Δy, and Δz as small

$$\Phi = \mu^3 r^{-2/7} \bar{f}(\bar{\zeta}) + \Delta\mu\mu^2 \, r^{-2/7}(3\bar{f} - \bar{\zeta}(d\bar{f}/d\bar{\zeta}))$$

$$-\Delta x \mu^2 (\gamma + 1)^{-1/3} \, r^{-6/7}(d\bar{f}/d\bar{\zeta})$$

$$+\mu^3 r^{-9/7}((2/7)\bar{f} + (4/7)\bar{\zeta}(d\bar{f}/d\bar{\zeta}) \; (\Delta y \cos\theta + \Delta z \sin\theta)$$

$$+ \ldots$$

or

$$\Phi = (\mu + \Delta\mu)^3 (r + \Delta r)^{-2/7} \, \bar{f}(\bar{\bar{\zeta}})$$

$$= \mu^3 \, r^{-2/7} \, \bar{f}(\bar{\zeta}) - \Delta\mu\mu^2 15 \, r^{-2/7} \sigma^{1/7} g^{(0)}(\sigma, -(2/7))$$

$$+ \Delta x \mu^2 (\gamma+1)^{-1/3} (8/3) r^{-6/7} \sigma^{3/7} g^{(0)}(\sigma, -(6/7))$$

$$- \mu^3 (16/3) r^{-9/7} \sigma^{8/7} g^{(1)}(\sigma, -(9/7))$$

$$(\Delta y \cos\theta + \Delta z \sin\theta) \qquad (88)$$

$$+ \ldots$$

These equations can be used to identify the modifications of a basic field (that is, the changes $\Delta\mu$, Δx, Δy, and Δz) as one proceeds from one iteration to the next one. Eqs. (86) and (87) are rewritten, using Eqs. (74), (29) and (31). One obtains from Eq. (86)

$$\Phi = (y - \Delta y)^{2/5}(\mu + \Delta\mu)^3\ \bar{f}(\bar{\bar{\zeta}}) \tag{89}$$

$$= y^{2/5}\mu^3\bar{f}(\bar{\zeta}) + \Delta\mu\mu^2\ 5\ G_{sym}(\sigma,2/5)\exp(2t/5)$$

$$+ \mu x\mu^2(\gamma+1)^{-1/3}\ G_{sym}(\sigma,-(2/5))\exp(-2t/5)$$

$$- \mu y\ ^3(1/2)\ G_{anti}(\sigma,-(3/5))\exp(-3t/5)$$

$$+ \ldots$$

and from Eq. (88)

$$\Phi = (\mu + \Delta\mu)^3(r + \Delta r)^{-2/7}\ \bar{f}(\bar{\bar{\zeta}}) \tag{90}$$

$$= \mu^3 r^{-2/7}\bar{f}(\bar{\zeta}) - \mu^2\ 15\ G^{(0)}(\sigma,-2/7)\exp(-2t/7)$$

$$- \mu^2(\gamma+1)^{-1/3}(8/3)G^{(0)}(\sigma,-6/7)\exp(-6t/7)$$

$$- \mu^3(16/3)\ G^{(1)}(\sigma,-9/7)\exp(-9t/7)(\Delta y\ \cos\theta+\Delta z\ \sin\theta)$$

The Wronskian of the functions G_1 and G_2 (for all cases) is defined by

$$W(G_1G_2) = W(s,\sigma) = (\partial G/\partial\sigma)G_2 - (\partial G_2/\partial\sigma)G_1 \tag{91}$$

The following relation is a consequence of the general theory of ordinary linear differential equation

$$a(\sigma)W(s,\sigma) = k(s)^{-1}$$

where $a(\sigma)$ is one of the coefficients occurring in the differential Eq. (64) for the functions G. Important is the fact that this product depends only upon s. The expression k(s) will appear in future formulae. For a derivation, one writes Eq. (64), once for G_1, a second time for G_2, multiplies the first equation by G_2, and subtracts from the result the second equation multipled by G_1. One obtains

$$\left[\frac{\partial}{\partial\sigma}(a(\sigma)\frac{\partial G_1}{\partial\sigma})\right] G_2 - \left[\frac{\partial}{\partial\sigma}(a(\sigma)\frac{\partial G_2}{\partial\sigma}\right] G_1 = 0$$

and, hence (92)

$$\frac{\partial}{\partial\sigma}\left[a(\sigma)\left(\frac{\partial G_1}{\partial\sigma} G_2 - \frac{\partial G_2}{\partial\sigma} G_1\right)\right] = 0$$

which leads directly to Eq. (90). To compute the function k(s), one needs the functions G_1 and G_2 and their first derivatives with respect to σ at one station σ, for instance at $\sigma = \sigma_L$. One has, because of Eqs. (74)

$$W(G_{sym,1}, G_{sym,2}) = (\sigma_L - \sigma)^{5s/3} W(\mathcal{G}_{sym,1}, \mathcal{G}_{sym,2})$$

$$W(G_{anti,1}, G_{anti,2}) = (\sigma_L - \sigma)^{5s/3} \sigma W(\mathcal{G}_{anti,1}, \mathcal{G}_{anti,2})$$

$$W(G_1^{(m)}, G_2^{(m)}) = (\sigma_L - \sigma)^{7s/5} \sigma^m W(\mathcal{G}_1^{(m)}, \mathcal{G}_2^{(m)}) \qquad (93)$$

Because of Eq. (78), one has

$$W(\mathcal{G}_1, \mathcal{G}_2) = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} W(\mathcal{G}_3, \mathcal{G}_2) \qquad (94)$$

This is now evaluated for the vicinity of $\sigma = \sigma_L$. One obtains using Eqs. (77)

$$W(\mathcal{G}_1, \mathcal{G}_2) = -\sigma_L^{-1} (1 - (\sigma/\sigma_L))^{c-a-b-1}(c-a-b) \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$$

$$= \sigma_L^{(a+b-c)} (\sigma_L - \sigma)^{c-a-b-1} \frac{\Gamma(c)\Gamma(1+a+b-c)}{\Gamma(a)\Gamma(b)} \qquad (95)$$

Using the values of c-a-b listed before Eq. (77), and the expressions $a(\sigma)$ to be found in Eqs. (29) and (30), specialized to the vicinity of $\sigma = \sigma_L$, one finds from Eq. (92)

$$k_{sym}(s) = \sigma_L^{-(1/3)-(5s/3)} \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(1+a+b-c)} \quad ; \quad \alpha = 0$$

$$k_{anti}(s) = \sigma_L^{-(4/3)-(5s/3)} \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(1+a+b-c)} \quad ; \quad \alpha = 0 \tag{95}$$

$$k^{(m)}(s) = \sigma_L^{-m-(7/5)-(7s/5)} \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(1+a+b-c)} \quad ; \quad \alpha = 1$$

where the pertinent values of $a(s)$, $b(s)$, and $c(s)$ are listed in Eq. (76). In Eqs.(79) through (82), the values of s for which a or b are nonpositive integers have been listed. For values of s in the vicinity of point s_p where either a or b is a nonpositive integer--k_2, one has

$$\Gamma(a) = \frac{1}{s - s_p} \frac{(-1)^{k_2}}{(da/ds)\Gamma(1+k_2)} + 0(1) \tag{96}$$

$$\Gamma(b) = \frac{1}{s - s_p} \frac{(-)^{k_2}}{(db/ds)\Gamma(1+k_2)} + 0(1) \tag{97}$$

With these formulae, the dominant part of $k(s)$ can be determined provided that none of the other factors vanishes.

Let s_p be a point at which, simultaneously

$$b = -k_2, \; k_2 = 0, 1, 2...$$

$$c-a-b = e = k_1, k_1 = 1, 2, ...$$

Let for neighboring points

$$b = -k_2 + (db/ds)\,(s-s_p) = -k_2 + \Delta b$$

$$c-a-b = e = k_1 + \Delta e$$

Then one obtains

$$\frac{\Gamma(b)}{\Gamma(1+a+b-c)} = \frac{\Delta e}{\Delta b}(-)^{k_1-k_2} \frac{\Gamma(k_1)}{\Gamma(1+k_2)}, \quad \begin{array}{l} k_1 = 1, 2, ... \\ k_2 = 0, 1, 2, ... \end{array} \tag{98}$$

$$\frac{\Gamma(b)}{\Gamma(1+a+b-c)} = \frac{1}{\Delta b} \frac{(-)^{k_2}}{\Gamma(1+k_2)} \quad \begin{array}{l} k_1 = 0 \\ k_2 = 0, 1, 2... \end{array} \tag{99}$$

One obtains from Eq. (78)

$$\mathfrak{g}_1 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,\mathfrak{g}_2 \quad ; \qquad \begin{array}{l} a = -k_2, \text{ or } b = -k_2 \\ k_2 = 0, 1, 2, \ldots \\ c-a-b \neq k_1 \\ k_1 = 1, 2, \ldots \end{array} \tag{100}$$

The formula is applicable also for $c-a-b = -k_1$, $k_1 = 1, 2, 3, \ldots$ $b = -k_2$, $k_2 = 0, 1, 2, \ldots$ (a case which does not occur), but then a limiting process is necessary, because $\Gamma(c-a) = \Gamma(-k_1+b) = \Gamma(-k_1-k_2)$.

If at a point $s = s_P$, $c-a-b = 0$, then one has $c-a = b$, $c-b = a$. Therefore,

$$\mathfrak{g}_2(\sigma,\ s_P) = \mathfrak{g}_3(\sigma,\ s_P) = F(a,b,1,1,\ 1-(\sigma/\sigma_L)),\ c-a-b = k_1=0 \tag{101}$$

If, simultaneously, $c-a-b = e = 0$, and $b = -k_2$, $k_2 = 0, 1,2,\ldots,$ then $\mathfrak{g}_1$, $\mathfrak{g}_2$, and $\mathfrak{g}_3$ are polynomials. One obtains

$$\hat{\mathfrak{g}}_1(\sigma,\ s_P) = (-)^{k_2}\,\Gamma(1+k_2)\ \frac{\Gamma(c)}{\Gamma(a)}\,\mathfrak{g}_2(\sigma,s_P), \qquad \begin{array}{l} c-a-b = 0 \\ k_2 = 0, 1,2,\ldots \end{array} \tag{102}$$

Eq. (102) is not the limiting form of Eq. (100).

Now we consider cases where $c-a-b = e = k_1, k_1 = 1, 2, \ldots$ and $b = -k_2$, $k_2 = 0, 1, 2,\ldots$ Specializing the second of Eqs. (77), we define for $k_2 < k_1$

$$\bar{g}_2(\sigma,\ s_P) = \lim_{\varepsilon\to 0} F\ (a,-k_2,\ 1-k_1+\ \varepsilon,1-(\sigma/\sigma_L))$$

For $\varepsilon \neq 0$, and, therefore, also in the limit $\varepsilon \to \Omega$, the function F is a polynomial which terminates with the power $(1-(\sigma/\sigma_L))^{k_2}$. Alternatively, we can write

$$\bar{g}_2(\sigma,\ s_P) = \sum_{i=0}^{k_2} \frac{\Gamma(a+i)}{\Gamma(a)}\ \frac{\Gamma(1+k_2)}{\Gamma(1+k_2-i)}\ \frac{\Gamma(k_1-i)}{\Gamma(k_1)}\ \frac{1}{\Gamma(i+1)}\ \left(1-\frac{\sigma}{\sigma_L}\right)^i$$

$$k_2 < k_1 \tag{103}$$

We define, for $k_2 \geq k_1$, as $\bar{\bar{g}}_2$, the same sum, but terminating with the power $k_1 - 1$

$$\bar{\bar{g}}_2(\sigma, s_P) = \sum_{i=0}^{k_1-1} \frac{\Gamma(a+i)}{\Gamma(a)} \frac{\Gamma(1+k_2)}{\Gamma(1+k_2-i)} \frac{\Gamma(k_1-i)}{\Gamma(k_1)} \frac{1}{\Gamma(i+1)} \left(1-\frac{\sigma}{\sigma_L}\right)^i ;$$

$$k_2 \geq k_1 \qquad (104)$$

One obtains

$$\mathcal{g}_1(\sigma, s_P) = \frac{\Gamma(c)\Gamma(k_1)}{\Gamma(k_1-k_2)\Gamma(k_1+a)} \bar{g}_2(\sigma, s_P), \quad \begin{array}{l} c-a-b = k_1,\ k_1 = 1, 2, \ldots \\ b = -k_2,\ k_2 = 0, 1, 2, \ldots \\ k_2 < k_1 \end{array} \qquad (105)$$

One obtains, for $k_1 \leq k_2$, from Eq. (77)

$$\mathcal{g}_3(\sigma, s_P) = (1-(\sigma/\sigma_L))^{k_1} F(k_1-k_2, k_1+a, 1+k_1, (1-(\sigma/\sigma_L)) \qquad (106)$$

This is a polynomial which begins with the power $(1-(\sigma/\sigma_L))^{k_1}$ and terminates with $(1-(\sigma/\sigma_L))^{k_2}$. Here one obtains

$$\mathcal{g}_1(\sigma, s_P) = (-)^{k_1-k_2} \frac{\Gamma(c)\Gamma(1+k_2)}{\Gamma(a)\Gamma(1+k_1)} \mathcal{g}_3, \quad \begin{array}{l} c-a-b = k_1,\ k_1 = 1, 2, \ldots \\ b = -k_2,\ k_2 = 0, 1, 2, \ldots \end{array} \qquad (107)$$

Notice that the coefficient of $\mathcal{g}_3$ is not the limit of the coefficient of $\mathcal{g}_3$ in Eq. (78).

Also, needed are expressions for $\mathcal{g}_2$ in the vicinity of points $s = s_P$ where $c-a-b = k_1$, $k_1 = 0, 1, 2, \ldots$ The case $k_1 = 0$ is covered by Eq. (101).

If $b \neq -k_2$, $k_2 = 0, 1, 2, \ldots$, one obtains

$$\hat{g}_2(\sigma,s) = \sum_{i=0}^{k_1-1} (-)^i \frac{\Gamma(a+i)}{\Gamma(a)} \frac{\Gamma(b+i)}{\Gamma(b)} \frac{\Gamma(k_1-i)}{\Gamma(k_1)} \frac{1}{\Gamma(i+1)} (1-(\sigma/\sigma_L))^i$$

$$(-)^{k_1} \sum_{i=k_1}^{\infty} \frac{d}{ds} \left\{ \frac{s-s_P}{\Delta e} \frac{\Gamma(a+i)}{\Gamma(a)} \frac{\Gamma(b+i)}{\Gamma(b)} \frac{\Gamma(1+\Delta e)}{\Gamma(k_1+\Delta e)} \frac{\Gamma(1-\Delta e)}{\Gamma(1+i-k_1-\Delta e)} \right\} \frac{1}{\Gamma(i+1)}$$

$$(1-(\sigma/\sigma_L))^i$$

$$(-)^{k_1} \frac{1}{s-s_P} \frac{1}{de/ds} \frac{\Gamma(a+k_1)}{\Gamma(a)} \frac{\Gamma(b+k_1)}{\Gamma(b)} \frac{1}{\Gamma(k_1)\Gamma(1+k_1)} \hat{g}_3(\sigma, s_P)$$

$$k = 1, 2, 3, \ldots$$

$$b \neq -k_2, \; k_2 = 0, 1, 2, \ldots$$

$$c-a-b = e = k_1 + \Delta e \tag{108}$$

All functions of s are to be evaluated for $s = s_P$. Only the general structure of this formula will be needed. For the evaluation of derivatives (not needed in the present context) we note the following formulae.

$$\frac{d}{da} \log \frac{\Gamma(a+i)}{\Gamma(a)} = \begin{cases} 0, & i = 0 \\ \frac{1}{a} + \frac{1}{a+1} \cdot \frac{1}{a+i-1}, & i = 1, 2, 3, \end{cases}$$

$$\frac{d}{ds} \frac{s-s_P}{\Delta e(s)} = \frac{-d^2\Delta e/ds^2}{2(d\Delta e/ds)^2}, \quad c-a-b = k_1 + \Delta e \tag{109}$$

In cases where, in addition, $b_2 = -k_2$, $k_2 = 0, 1, 2, \ldots$ one has

$$\hat{g}_3(\sigma, s_P) = (1-(\sigma/\sigma_L))^{k_1} F(k_1+a, k_1-k_2, 1+k_1, (1-(\sigma/\sigma_L)) \tag{110}$$

For $k_1 > k_2$, this is an infinite series, for $k_1 \leq k_2$, it is a polynomial. One has

$$\lim_{s\to s_p} \hat{g}_2(\sigma,s) = \bar{g}_2(\sigma,s_p)$$

$$+ (-)^{k_1-k_2} \lim_{s\to s_p} \left(\frac{\Delta b}{\Delta e}\right) \frac{\Gamma(a+k_1)\Gamma(1+k_2)\Gamma(k_1-k_2)}{\Gamma(a)\Gamma(k_1)\Gamma(1+k_1)} \hat{g}_3(\sigma,s_p)$$

$$b = -k_2 + \Delta b,\ c-a-b = e = k_1 + \Delta e,\ k_2 < k_1$$

$$(111)$$

The function $\bar{g}_2$ is defined in Eq. (103). The factor of $\hat{g}_3$ is bounded. Finally,

$$\hat{g}_2(\sigma,s) = \bar{\bar{g}}_2 + \sum_{i=k_1}^{k_2} (-)^{k_1-i} \frac{d}{ds}\left(\frac{s-s_p}{\Delta e} \frac{\Gamma(a+i)}{\Gamma(a)} \frac{\Gamma(1+\Delta e)}{\Gamma(k_1+\Delta e)} \frac{\Gamma(1-\Delta e)}{\Gamma(1+i-k_1-\Delta e)} \frac{\Gamma(1+k_2-\Delta b)}{\Gamma(1+k_2-\Delta b-i)}\right)$$

$$\frac{1}{\Gamma(1+i)} (1-(\sigma/\sigma_L))^i$$

$$+ \frac{1}{s-s_p} \frac{1}{de/ds} \frac{\Gamma(a+k_1)}{\Gamma(a)} \frac{\Gamma(1+k_2)}{\Gamma(k_1)\Gamma(1+k_2-k_1)\Gamma(1+k_1)} \hat{g}_3(\sigma,s_p)$$

$$+ (-)^{k_2-k_1} \frac{\Delta b}{\Delta e} \frac{\Gamma(1+k_2)}{\Gamma(k_1)} \sum_{i=k_{2+1}}^{\infty} \frac{\Gamma(a+i)}{\Gamma(k_1)} \frac{\Gamma(i-k_2)}{\Gamma(1+i-k_1)} \frac{1}{\Gamma(i-1)} (1-(\sigma/\sigma_L))^i$$

$$k_1 = 1, 2, \ldots, k_2 = 0, 1, 2\ldots$$

$$k_2 > k_1 \qquad (112)$$

SECTION VI
EVALUATION OF THE LAPLACE TRANSFORM

Now we return to Eq. (59). Let G_1 be a particular solution of the homogeneous part of this equation (that is of Eq. (64)) which satisfies the conditions at $\sigma = 0$. The definition is found in Eqs. (74).

The boundary condition at $\sigma = \sigma_L$ is derived from the requirement that in the original region of definition of the Laplace transform $\tilde{\psi}^{(m)}$ is bounded (see the discussion preceeding Eq. (63)).

The behavior of the function $\psi^{(m)}$ is partially determined by the inhomogeneous part $r^{(m)}$ defined in Eq. (60)). Assume momentarily that $r^{(m)} \equiv 0$ in the vicinity of $\sigma = \sigma_L$. In this region $\psi^{(m)}$ is then given by a combination of two linearly independent functions G, of G_2 and G_3, say. According to Eq. (66) $G = h^{-s}g$, therefore $\tilde{\psi}^{(m)}$(see Eq.(62))is a linear combination of g_2 and g_3. In the original region of definition of the Laplace transform (given by the inequalities (63)), the function g_3 tends to infinity at $\sigma = \sigma_L$ (this is seen from Eqs. (89) and (77)). At $\sigma = \sigma_L$ the particular solution G_3 is therefore not allowed to be present. On this basis one finds for the solution of Eq. (59)

$$\psi^{(m)}(s,\sigma) = k(s) \; [G_1(s,\sigma) \int_{\tilde{\sigma}=\sigma_L}^{\sigma} r^{(m)}(s,\sigma) \; G_2(s,\tilde{\sigma})d\tilde{\sigma}$$

$$-G_2(s,\sigma) \int_{\tilde{\sigma}=0}^{\sigma} r^{(m)}(s,\tilde{\sigma})G_1(s,\tilde{\sigma})d\tilde{\sigma}] \tag{113}$$

The conditions at $\sigma = 0$ and $\sigma = \sigma_L$ are satisfied by the choice of the lower limits of the integrals. In the first place this expression holds in the region of definition of the Laplace transform. By analytical continuation it is then extended throughout the complex s plane.

Here the expression $r^{(m)}$ (Eq. (60)) is substituted. This expression consists of three summands. For the second summand an integration by parts is carried out. The contributions of the terms at the limits of the integrals $\sigma = 0$, and $\sigma = \sigma_L$ vanish, the first one always, the second one for values of s in the original region of definition of the Laplace transform. With some rearrangement of the terms one then obtains

$$\psi^{(m)}(s,\sigma) = k(s) \Big\{ G_1^{(m)}(s,\sigma) K(\phi^{(m)}, G_2^{(m)}, \sigma_L, \sigma) -$$

$$G_2^{(m)}(s,\sigma) \; K(\phi^{(m)}, G_1^{(m)}, 0, \sigma) \Big\} \tag{114}$$

where

$$K(\phi^{(m)}, G^{(m)}, a, b)$$

$$= \int_a^b \exp(-st_s(\tilde{\sigma})) \Big\{ \phi^{(m)} [(\beta + s) b(\tilde{\sigma}) G^{(m)} + (dt_s/d\tilde{\sigma}) a(\tilde{\sigma}) (\partial G^{(m)}/\partial\tilde{\sigma})]$$

$$+ G^{(m)} [b(\tilde{\sigma}) \phi_t^{(m)} - a(\tilde{\sigma}) (dt_s/d\sigma) \phi_{\tilde{\sigma}}^{(m)}] \Big\} d\tilde{\sigma} \tag{114a}$$

Here the arguments of $\phi^{(m)}$ and its derivatives are $t_s(\tilde{\sigma})$ and σ. The arguments of $G^{(m)}$ and its derivatives are $\tilde{\sigma}$ and s. We set

$$\bar{\omega}^{(m)}(t,\sigma,s) = G^{(m)}(s,\tilde{\sigma}) \; \exp(-(\beta + s)t) \tag{115}$$

$$\omega^{(m)}(t,\sigma,\theta,s) = \bar{\omega}^{(m)}(t,\sigma,s) \; \exp(-im\theta) \tag{116}$$

The function $\omega^{(m)}$ is, according to Eq. (69), a solution of the original partial differential equation (37) in the t, σ, θ space. Expressing $\phi^{(m)}$ in terms of ϕ by Eq. (53), one then obtains

$$K(\phi^{(m)},G^{(m)},a,b) = -(1/2\pi)\int_{\theta=0}^{2\pi}\int_{a}^{b} \exp(\beta t)\Big\{b(\tilde{\sigma})\ [\phi\omega_t^{(m)} - \phi_t\omega^{(m)}]d\tilde{\sigma}$$

$$- a(\tilde{\sigma})[\phi\omega_{\tilde{\sigma}}^{(m)} - \phi_{\tilde{\sigma}}\omega^{(m)}]dt\Big\}d\theta \qquad (117)$$

Here it is assumed that the path of integration from a to b in the $\tilde{\sigma}$,t plane is given by $t = t_s(\tilde{\sigma})$. The subscript s of t has been omitted; dt stands for $(dt_s/d\tilde{\sigma})d\tilde{\sigma}$.

The integral K has the form of Eq. (44) and Eq. (44) is related to the expression (40), (41) and (42) by Eqs. (45). Accordingly, K can be evaluated in the physical space. For a closer study we express $\omega^{(m)}$ as far as possible in physical coordinates. According to Eqs. (21) and (22), one has

$$\exp t = yh, \quad \alpha = 0$$

$$\exp t = rh, \quad \alpha = 1$$

With Eqs. (29) through (32), and (74) one then obtains

$$\bar{\omega}_{sym} = y^{(1/5)-s}\ \sigma^{-(1/10)+(s/2)}\ (\sigma_L-\sigma)^{(5s/3)-(1/6)}\ \hat{g}_{sym}(\sigma)$$

$$\bar{\omega}_{anti} = y^{(1/5)-s}\ \sigma^{(2/5)+(s/2)}\ (\sigma_L-\sigma)^{(5s/3)-(1/6)}\ \hat{g}_{anti}(\sigma)$$

$$\bar{\omega}^{(m)} = y^{-(4/7)-s}\ \sigma^{(m/2)+2/7)+(s/2)}\ (\sigma_L-\sigma)^{(7s/5)+(2/5)}\ \hat{g}^{(m)}(\sigma)$$

$$(118)$$

These formulae express $\bar{\omega}$ in terms of y and σ. At $\sigma = \sigma_L, \zeta$ is a regular function of σ, and ζ is expressed by coordinates in the physical plane. It follows that in the vicinity of the curve $\sigma = \sigma_L$ and for $y \neq 0$ or $r \neq 0$, there is a one to one mapping from the σ,y plane to the x, y or x, r plane. The singular behavior of the functions $\bar{\omega}$ at $\sigma = \sigma_L$ which is evident in Eqs. (118), therefore carries over to the physical plane. The integration in the expression K (in the form of Eqs. (40), (41), or (42)) will be extended over the outer boundary surface S of the computed flow field. One can therefore choose x, y (or r) as functions which are regular at $\sigma = \sigma_L$. The function ϕ, ϕ_t and ϕ_σ will be regular, unless a singularity, induced by the shape of the body, propagates along the limiting characteristic. Singularities in the integrands of the expressions (40), (41), and (42) therefore appear solely because of singularities in $\bar{\omega}$. The functions $\mathfrak{G}_2$ are regular at $\sigma = \sigma_L$. If one substitutes those into Eqs. (119), then one finds that the strongest singularities in Eqs. (40), (41), and (42) those caused by ω_x and ω_y have the form $(\sigma_L-\sigma)^{(5s/3) - (7/6)}$ for $\alpha = 0$, and $(\sigma_L-\sigma)^{-(3/5)+(7s/5)}$ for $\alpha = 1$. If, on the other hand, the functions ω are formed with $\mathfrak{G}_3$, (Eq. (77)), then the functions ω are regular because of the factor $(1-(\sigma/\sigma_L))^{c-a-b}$ in $\mathfrak{G}_3$. The same result can be found, from the expression (117) in the $\tilde{\sigma}$, t plane, but the discussion is quite cumbersome.

Poles of the expression (114) arise for several reasons:

a. In the region of the s plane where $G_2^{(m)}$ is the dominant solution, $G_2(s,\sigma)$ will have poles at those points where the difference of the exponents c-a-b is a positive integer.

b. Poles may arise in the analytic continuation of the function $K(\phi^{(m)}, G_2, \sigma_L, \sigma)$.

c. The function k(s) may have poles.

To study the poles which arise according to reason b, we develop the integrand in powers of $(\sigma_L - \sigma)$. In general these will be fractional powers, which depend upon s. Consider an expression

$$\int_{\tilde{\sigma}=\sigma}^{\sigma} (\sigma-\sigma_L)^{p(s)} d\tilde{\sigma} = \frac{1}{p(s)+1} (\sigma-\tilde{\sigma}_L)^{p(s)+1} \Big|_{\tilde{\sigma}=\sigma_L}^{\sigma} \tag{119}$$

Evaluating the expression in the region where the integral exists, that is for $p(s) = -1$, one obtains

$$\int_{\tilde{\sigma}=\sigma_L}^{\sigma} (\tilde{\sigma}-\sigma_L)^{p(s)} d\tilde{\sigma} = \frac{1}{p(s)+1} (\sigma-\sigma_L)^{p(s)+1} \tag{120}$$

The right hand side can be continued analytically throughout the s plane. A pole arises at $p(s) = -1$. Assume that this happens at a point $s = s_p$. Then one has in the vicinity of such a pole

$$\int_{\tilde{\sigma}=\sigma_L}^{\sigma} (\tilde{\sigma}-\sigma_L)^{p(s)} d\tilde{\sigma} = \frac{1}{(s-s_p)} \cdot \frac{1}{(dp(s)/ds)}\Big|_{s=s_p} + 0(1) \tag{121}$$

We have found that the powers which arise in $K(\phi^{(m)}, \omega_2^{(m)}, \sigma_2, \sigma)$ are $(\sigma_L-\sigma)^{(5s/3)-(7/6)}$ for $\alpha = 0$, and $(\sigma_L-\sigma)^{(7s/5)-(3/5)}$ for $\alpha = 1$. Poles may therefore arise if, for $\alpha = 0$

$$-(7/6) + (5s/3) = -k_3, \qquad k_3 = 1, 2, 3, \ldots$$

that is for

$$s = -(3k_3/5) + (7/10)$$

For these values one has

$$c-a-b = k_3-1$$

The same result is found for $\alpha = 1$. Actually, the value $k_3 = 1$ must be excluded. For then $c-a-b = 0$. In this case $\vartheta_2 = \vartheta_3$, according to Eq. (101). But we have found that no singularity will occur if ω is formed with ϑ_3. In summary, poles will arise in the analytic continuation of K for $c-a-b = k_1$; $k_1 = 1,2,\ldots.$

In the above discussion we have excluded the possibility that the development of ϕ, ϕ_t or ϕ_σ contains nonintegral powers of $(\sigma-\sigma_L)$. If this possibility is admitted, then further poles will arise in the function K. The particular solution for the values s pertaining to these poles, give individually the propagation of a singularity along the limiting characteristic. This possibility is included in the discussions of Reference 4.

If ϕ, ϕ_t and ϕ_σ are regular at $\sigma=\sigma_L$, then poles in the function $\mathfrak{g}_2$ (reason a) and poles in K occur at the same values of s. The singularities in $\mathfrak{g}_2$ have the form $(s-s_1)^{-1}$ const $\mathfrak{g}_3$. We have seen above that the functions $\mathfrak{g}_3$ do not cause singularities in K. It follows that no poles of second order will arise by the coincidence of poles in $\mathfrak{g}_2$ and in K.

The function k(s) (Eq. 96)) vanishes because of the factor $(\Gamma(1-(c-a-b)))^{-1}$ for the values of s for which $c-a-b = k_1; k_1 = 1,2,\ldots$, unless simultaneously $a = -k_2$ or $b = -k_2$, $k_2 = 0,1,2\ldots$ This factor cancels the effects of the poles which occur in $\mathfrak{g}_2$ and in $K(\phi, G_2, \sigma_L, \sigma, s)$. If $\alpha = 0$, then the values of s, for which $c-a-b = k$, and $b = -k_2$ or $a = -k_2$, are always different. The only poles which can arise are due to k(s). It has been demonstrated in the discussion following Eq. (83), that for $\alpha = 1$, there are some values of s for which simultaneous $c-a-b = k_1$ and $b = k_2$. For these poles separate formulae must be derived.

Poles caused by the factor k(s) in Eq. (114) arise, according to Eq. (96), if $a = -k_2$ or $b = -k_2$, $k_2 = 0, 1, 2\ldots$ provided that $c-a-b \neq k_1$, $k_1 = 1, 2, 3\ldots$ One recognizes that one has for all possible poles either $a = -k_2$ or $b = -k_2$; this implies that $\mathfrak{g}_1$ is a polynomial in σ. This guarantees that $\mathfrak{g}_1$ is regular at $\sigma=\sigma_L$. That particular solutions of only this kind will be encountered must be expected because of the assumption that ϕ, ϕ_x and ϕ_y are free of singularities at $\sigma=\sigma_L$.

One has, according to Eq. (100), for these cases

$$\frac{G_1}{G_2} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad , \quad k_1 \neq 0, 1, 2... \tag{122}$$

$$\frac{G_1}{G_2} = (-)^{k_2} \frac{\Gamma(c)}{\Gamma(a)} \quad , \quad k_1 = 0 \tag{123}$$

These ratios can be evaluated for the pertinent values of $s = s_1$. The specific values for $\alpha = 0$ are found in Eqs. (79), (80) and (81). For $\alpha = 1$, they are computed from Eq. (82). In these cases, Eq. (115) simplifies to the following expression

$$\psi^{(m)}(s,\sigma) = \frac{1}{(s-s_p)(d(k^{-1})/ds)} \frac{G_1}{G_2} G_2^{(m)}(\sigma,s_p)K(\phi,\omega_2^{(m)}(t,\sigma,\theta,s_p),0,\sigma_L) \tag{124}$$

In the expression K, defined in Eq. 117), we have written $\omega_2^{(m)}$ instead of the argument $G_2^{(m)}$, which might have been used just as well, to emphasize the significance of the particular solutions $\omega_2^{(m)}$. In these cases, the two integrals in Eq. (114) can be combined. (The limits of the integral become then 0 and σ_L. In the vicinity of a point $s = s_p$ the σ-dependence of $\psi^{(m)}(s,\sigma)$ is solely introduced by $G_2^{(m)}(\sigma,s_p)$. The ratio G_1/G_2 is computed from Eqs. (122) and (123).

In cases where at a point $s = s_p$ one has simultaneously $b = k_2$, $k_2 = 0, 1, 2...$ and $c-a-b = e = k_1$, $k_1 = 1, 2, 3...$ one obtains a finite limiting value for k(s), which can be evaluated by Eq. (98). We distinguish between cases, where $k_2 < k_1$ and $k_2 > k_1$.

If $k_2 < k_1$, then, according to Eq. (102), the function $\mathfrak{g}_2$ remains finite in the limit $s \to s_p$. In this limit $\mathfrak{g}_2$ consists of the polynomial $\bar{g}_2$, and the function $\mathfrak{g}_3$ multiplied by a constant. Poles arise in K, because of $\bar{g}_2$.

The behavior of $\psi^{(m)}(s,\sigma)$ in the vicinity of s_p is computed by applying Eq. (122) to the expression (118), substituting for $\omega^{(m)}$ the expression (117) formed with $\bar{g}_2$. The procedure is rather complicated because it is based on a series development of the integrand. If it should be needed, one would probably carry it out in the physical plane, on the basis of Eq. (40), (41) or (42). At the moment we simply write

$$K(\sigma,s) = \frac{1}{s-s_p} \text{ Residual of } K(\phi, \bar{\omega}_2, \sigma_L, \ldots) \text{ at } s = s_p$$

(The second limit of the integral is unimportant.) Then, one has, using Eq. (105)

$$\psi^{(m)}(s,\sigma) = \frac{1}{s-s_p} k(s_p) \frac{\Gamma(c)\Gamma(k_1)}{\Gamma(k_1-k_2)\Gamma(k_1+a)} \bar{G}_2(\sigma,s_p)$$

$$\times \text{ Residual of } K(\phi, \bar{\omega}_2^{(m)}, \sigma_L, \ldots) \text{ at } s = s_p,\ k_2 < k_1 \qquad (125)$$

The function $\mathfrak{g}_2$ consists for $k_2 \geq k_1$ according to Eq. (112), (a) of a polynomial $\bar{g}_2$ which contains powers of $(1 - (\sigma/\sigma_L))$ up to k_1-1 with finite coefficient, (b) of a pole whose σ dependence is given by the polynomial $\mathfrak{g}_3$ (in which k_2 is the maximum power of $(1-\sigma/\sigma_L)$, and (c) an infinite series starting with $(1-\sigma/\sigma_L)^{k_1}$. A pole arises because of the contributions a and b. Invoking Eq. (107) one obtains

$$\psi^{(m)}(s,\sigma) = \frac{1}{s-s_p} k(s_p)(-)^{k_1-k_2} \frac{\Gamma(c)\Gamma(1+k_2)}{\Gamma(a)\Gamma(1+k_1)} G_3(\sigma,s_p) \times$$

$$\Big\{\text{Residual of } K(\phi, \bar{\bar{\omega}}_2^{(m)}, \sigma_L \ldots) \text{ at } s = s_p)$$

$$- \frac{1}{de/ds} \frac{\Gamma(a+k_1)\Gamma(1+k_2)}{\Gamma(a)\Gamma(k_1)\Gamma(1+k_2-k_1)\Gamma} K(\phi, \omega_3^{(m)}, 0, \sigma_L, s_p)\Big\} \qquad (126)$$

SECTION VII
THE SOLUTION IN THE PHYSICAL PLANE

In Eq. (52) a Fourier decomposition with respect to θ of the solution of the boundary value problem formulated in Section III has been made. The solution for one Fourier component, expressed by means of its Laplace transform is given by

$$\phi^{(m)}(t,\sigma) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \psi^{(m)}(s,\sigma)\ \exp(st)ds \tag{127}$$

To ensure the behavior postulated in Eqs. (47) and (48), the path of integration must be chosen somewhat to the right of the lines Re $s = (2/5)$ for $\alpha = 0$, and $\mathrm{Re}(s) = -(2/7)$ for $\alpha = 1$. How far this path can be shifted to the right depends upon the location of the neighboring poles in the s-plane.

The integrand of Eq. (127) depends upon s. The integrand becomes small for negative t if Re s is large and positive. We close the path of integration by a sequence of circles in the right half of the complex plane whose radius goes to infinity. It is shown in Ref. [4] that in a certain well defined region in the left half of the t,σ plane the contribution to the integral due to the integration along these circles vanishes in the limit of an infinite radius. In this part of the t,σ plane the functions $\phi^{(m)}(t,\sigma)$ can then be represented in terms of the residuals at the poles of $\psi^{(m)}(s,\sigma)$. With respect to these poles one travels in the counterclockwise direction.

If the far field conditions are satisfied, then, according to the discussions of Section II following Eq. (48), $\phi^{(m)} \equiv 0$ for $t < t_s(\sigma)$. The region in which the representation of $\phi^{(m)}$ in terms of the poles in the right half of the complex s plane is valid, lies within the region $t \leq t_s(\sigma)$. In order for $\phi^{(m)}$ to be identically equal to zero for $t < t_s(\sigma)$ it is necessary that residuals at all of these poles vanish. This observation is decisive for the formulation of the far field conditions.

The path of integration can also be closed by circles in the left half of the s plane and the contribution of these circles vanish for a well defined region in the right half of the t,σ plane. In the present context one is not interested in a representation of $\phi^{(m)}(t,\sigma)$ in terms of the residuals, except for the contribution of those poles, which give a modification of the basic flow field.

Expressions for $\psi^{(m)}(s,\sigma)$ valid in the vicinity of the poles have been derived in Section VI. In the vicinity of a pole with $s = s_i$ they have the form

$$\frac{1}{s - s_i} c_{i,m} G_2^{(m)}(\sigma,s_i)$$

where $c_{i,m}$ is a constant.

Closing the path of integration by circles in the right half of the complex s plane, one obtains

$$\phi^{(m)}(t,\sigma) = - \sum_i c_{i,m} G_2^{(m)}(\sigma,s_i) \exp(s_i t) \tag{128}$$

and

$$\phi(t,\sigma,\theta) = -\sum_{m=-\infty}^{m=\infty} \sum_i c_{im} G_2^{(m)}(\sigma,s_i, \exp(s_i,t) \exp(im\theta) \tag{129}$$

The summation must be extended over all poles to the right of the lines

$$\begin{aligned} \text{Res} &= 2/5 \quad \text{for} \quad \alpha = 0 \\ \text{Res} &= -2/7 \quad \text{for} \quad \alpha = 1 \end{aligned} \tag{130}$$

The individual terms are particular solutions of the partial differential equation (37), which is equivalent to Eqs. (12) or (13).

The inner sum converges, according to Ref. 2 in that region of the t,σ plane for which the closing of the path of integration in the complex s plane is permissible. This region usually does not extend to the curve $t = t_s(\sigma)$ (which is the map of the outer surface s of the computed flow field). In order to express the postulate that $\phi \equiv 0$ to the left of $t < t_s(\sigma)$, it suffices if this condition is imposed in a smaller region. One obtains the condition $c_{i,m} = 0$ for all m and all poles to the right of the lines characterized above. This leads to

$$K(\phi,\ \omega_2^{(m)}(t,\ \sigma,\ \theta,\ s_i),\ o,\ \sigma_L) = 0 \tag{131}$$

where K is defined in Eq. (117). The notation used here is the same as in Eq. (124). K is best evaluated in the physical plane, by means of Eq. (40), (41) or (42). The expressions ω_2 are defined in Eqs. (118). Here the function $\mathfrak{g}_2$, or also the functions $\mathfrak{g}_1$, which for these poles are proportional to $\mathfrak{g}_2$, must be substituted. They are defined by Eqs. (77).

One needs for the evaluation in the physical plane

$$\frac{\partial\sigma(x,y)}{\partial x} = \mu^{-1}(10/3)(\gamma+1)^{-1/3}(\sigma/y^2)^{2/5}(\sigma+(4/3))^{-1}\sigma \qquad \alpha = 0$$

$$\frac{\partial\sigma(x,y)}{\partial y} = (4/3)y(\sigma/y^2)(2-\sigma)(\sigma+4/3)^{-1} \qquad \alpha = 0$$

$$\tag{132}$$

$$\frac{\partial\sigma(x,r)}{\partial x} = \mu^{-1}(7/10)(\gamma+1)^{-1/3}(\sigma/r^2)^{2/7}(\sigma+(1/5))^{-1}\sigma \qquad \alpha = 1$$

$$\frac{\partial\sigma(x,r)}{\partial r} = (2/5)r(\sigma/r^2)(1-2\sigma)(\sigma+(1/5))^{-1} \qquad \alpha = 1$$

One has

$$\sigma/y^2 = (\gamma+1)^{5/6}\ \mu^{5/2}\ (-x)^{5/2}\ (2-\sigma)^{5/2} \qquad \alpha = 0$$

$$\tag{133}$$

$$\sigma/r^2 = (\gamma+1)^{7/6}\ \mu^{7/2}\ (-x)^{7/2}\ (1-2\sigma)^{7/2} \qquad \alpha = 1$$

These expressions are used in a vicinity of $\sigma = 0$.

$$\frac{\partial\omega_{sym}}{\partial x} = \mu^{-1}(10/3)(\gamma+1)^{-1/3}(\sigma/y^2)^{(s_P/2)+(3/10)}$$

$$(\sigma+(4/3))^{-1}(\sigma_L-\sigma)^{(5s_P/3)-(7/6)}$$

$$\Big\{[(s_P/2)-(1/10)](\sigma_L-\sigma)-((5s_P/3)-(1/6))\ \sigma]\mathcal{G}\ (\sigma,s_P)$$

$$+\ \sigma(\sigma_L-\sigma)d\mathcal{G}/d\sigma\Big\}$$

$$\frac{\partial\omega_{anti}}{\partial x} = \mu^{-1}(10/3)(\gamma+1)^{-1/3}(\sigma/y^2)^{((s_P/2)+(3/10))}$$

$$(\sigma+(4/3))^{-1}(\sigma^{1/2}(\sigma_L-\sigma)^{(5s_P/3)-(7/6)}$$

$$\Big\{[(s_P/2)+(2/5)](\sigma_L-\sigma)-((5s_P/3)-(1/6))\sigma]\mathcal{G}+\sigma(\sigma_L-\sigma)(d\mathcal{G}/d\sigma)\Big\}$$

$$\partial(\omega_{sym})/\partial y = y(\sigma/y^2)^{(s_P/2)+(9/10)}(\sigma_L-\sigma)^{(5s_P/3)-(7/6)}(\sigma+(4/3))^{-1}$$

$$\Big\{[-(10/3)((s_P/2)-(1/10))(\sigma_L-\sigma)-(5s_P/3)-(1/6))(4/3)(2-\sigma)]\mathcal{G}(\sigma,s_P)$$

$$+(4/3)(2-\sigma)(\sigma_L-\sigma)(\partial\mathcal{G}/\partial\sigma)\Big\} \tag{134}$$

$$\partial(\omega_{anti})/\partial y = (\sigma/y^2)^{(s_P/2)+(2/5)}(\sigma_L-\sigma)^{(5s_P/3)-(7/6)}(\sigma+(4/3))^{-1}$$

$$\Big\{[-(10/3)((s_P/2)-(1/10))(\sigma_L-\sigma)\sigma-((5s_P/3)-(1/6))(4/3)(2-\sigma)\sigma$$

$$-(2/3)(2-\sigma)(\sigma_L-\sigma)]\mathcal{G}+(4/3)(2-\sigma)(\sigma_L-\sigma)\sigma(d\mathcal{G}/d\sigma)\Big\}$$

$$(\partial\omega^{(m)}/\partial x) = \mu^{-1}(7/10)(\gamma+1)^{-1/3}\sigma^{m/2}(\sigma/r^2)^{(s_P/2)+(4/7)}$$

$$(\sigma+(1/5))^{-1}(\sigma_L-\sigma)^{(7s_P/5)-(3/5)}$$

$$\{[((m/2)+(s_P/2)+(2/7))(\sigma_L-\sigma)-((7s_P/5)+(2/5))\sigma]\hat{g}(\sigma,s_P)$$

$$+\ \sigma(\sigma_L-\sigma)(d\hat{g}/d\sigma)\}\exp(-im\theta) \qquad (135)$$

One has

$$\partial\omega^{(m)}/\partial y = \frac{\partial\omega^{(m)}}{\partial r}\cos\theta + im\frac{\omega^{(m)}}{r}\sin\theta$$

$$\partial\omega^{(m)}/\partial z = \frac{\partial\omega^{(m)}}{\partial r}\sin\theta - im\frac{\omega^{(m)}}{r}\cos\theta \qquad (136)$$

Here

$$\partial\omega^{(m)}/\partial r = \exp(-im\theta)\sigma^{(m/2)-1/2}(\sigma/r^2)^{(s_P/2)+(11/14)}$$

$$(\sigma_L-\sigma)^{(7s_P/5)-(3/5)}(\sigma+(1/5))^{-1} \qquad (137)$$

$$\Big\{[(\sigma_L-\sigma)(m/5)(1-2\sigma)-(14/5)((s_P/2)+(2/7))\sigma) -$$

$$((7s_P/5)+(2/5))(1-2\sigma)\sigma]\hat{g}+(1-2\sigma)\sigma(\sigma_L-\sigma)\sigma d\hat{g}/d\sigma\Big\}$$

for m = 0, this reduces to

$$\frac{\partial\omega^{(0)}}{\partial r} = \sigma^{1/2}(\sigma/r^2)^{(s_P/2)+(11/14)}(\sigma_L-\sigma)^{(7s_P/5)-(3/5)}(\sigma+(1/5))^{-1}$$

$$\Big\{[(\sigma_L-\sigma)(-(14/5)(s_P/2)+(2/7))-(7s_P/5)+(2/5)(1-2\sigma)]\hat{g}(0)$$

$$+\ (1-2\sigma)(\sigma_L-\sigma)d\hat{g}^{(0)}/d\sigma\Big\} \qquad (138)$$

The values of s_p are given by Eq. (79) and (81). For symmetric solutions, one has

$$s_p = (2/5) + (6k/5), \qquad k = 1, 2, \ldots \tag{139}$$

The value k = 0, which is included in Eq. (79) must be omitted because it lies outside of the region given by Eq. (130), it amounts to a change of the intensity of the basic singularity, and such a change must, of course, be admitted. For antisymmetric solutions, one has

$$s_p = 1 + (6k/5) \qquad k = 0, 1, 2, \ldots \tag{140}$$

The values of s for $\alpha = 1$ are computed from Eq. (82). The value for m = 0 and k = 0 must be omitted because they lie outside (although at the border) of the region characterized by Eq. (130). According to Eq. (100), which is valid for these poles, the function $\mathfrak{g}_2$ is proportional to $\mathfrak{g}_1$. As long as the pole strength is set equal to zero, the factor of proportionality is, of course, unimportant. It is best to compute $\mathfrak{g}_2$ from the first of Eqs. (77).

There exists the possibility to extend the present approach to boundary conditions for a free stream Mach number close to one. In such cases one would need the actual strength of the poles. This (and other factors) have therefore been carried in the analysis.

Assume that a correction to the flow field has been computed with the use of the far field conditions Eqs. (131). As a second step, one adjusts (if necessary) the intensity and the origin for the basic singularity. This is done on the basis of Eqs. (89) and (90), which express the effect of a change of μ and of the location of the origin by means of certain particular solutions.

For this purpose, one closes the curve of the original inversion integral by circles in the left half of the complex s plane. One then obtains

$$\phi^{(m)}(t,\sigma) = \sum_i c_{im}\, G_2^{(m)}(\sigma, s_i)\, \exp(s_i t) \tag{141}$$

The summation is over all poles that lie to the left of the line given by Eqs. (130) or on this line. The expression is slightly different for poles where simultaneously $c-a-b = k_1$, $k_1=1,\ldots$ and $b = -k_2$, $k_2 = 0, 1, 2,\ldots$ But in the present discussion these poles are of no interest. The values of $c_{i,m}$ equal to the residuals at the individual points, which are given by Eq. (124), (125) and (126). The adjustment of the basic solution is based on Eqs. (89) or (90).

This requires for $\alpha = 0$ that we find the contributions of $G_{sym}(\sigma, 2/5)$, $G_{sym}(\sigma,-2/5)$ and $G_{anti}(\sigma-3/5)$. The corresponding contributions for $\alpha = 1$ are $G^{(0)}(\sigma, -2/7)$, $G^{(0)}(\sigma,-(6/7))$, and $G^{(1)}(\sigma, -9/7)$. Let us denote the constants $c_{i,m}$ in the same manner as the corresponding functions G. These constants can be evaluated numerically. Combining these expressions with the basic flow, one obtains an expression (for $\alpha = 0$)

$$
\begin{aligned}
\tilde{\Phi}_0 = y^{2/5}\, \overline{f}(\zeta) &+ c_{sym,2/5}\, G_{sym}(\sigma, 2/5) \exp(2t/5) \\
&+ c_{sym,-2/5}\, G_{sym}(\sigma,-2/5) \exp(-2t/5) \\
&+ c_{anti,-3/5}\, G_{anti}(\sigma,-3/5) \exp(-3t/5)
\end{aligned}
$$

According to Eq. (89) this is an approximation to

$$\overline{\Phi}_0 = (y - \Delta y)^{2/5} (\mu + \Delta\mu)^3\, \overline{f}(\overline{\overline{\zeta}}) \tag{142}$$

(with $\overline{\overline{\zeta}}$ defined in Eq.(85)) if one sets

$$
\begin{aligned}
\Delta\mu &= (1/5)\mu^{-2} c_{sym,2/5} \\
\Delta x &= (\gamma + 1)^{1/3} \mu^{-2} c_{sym,-2/5} \\
\Delta y &= -2\mu^{-3} c_{anti,-3,5}
\end{aligned}
\tag{143}
$$

A similar procedure is applied for $\alpha = 1$.

In the following iteration step the expression $\bar{\phi}$ is then used for the basic flow field. This correction needs to be carried out, only if $\Delta\mu$, Δx, and Δy are fairly large. Sometimes it may suffice, if one makes a correction only for the dominant term, which is given by $\Delta\mu$.

SECTION VIII
NUMERICAL IMPLEMENTATION

It is assumed that an approximation to the flow field is available and that one wants to improve it iteratively. One needs for the application of the far field a characterization of the basic field for the outer region by the value of μ and the coordinates of the origin to which it refers. The correction to the flow field is then computed using Eqs. (131) as conditions to be imposed at the distant boundary of the computed flow field. For the field so obtained determines, in a second step, corrections to the basic field.

The computed part of the flow field will frequently consist of one portion which is given by the result of the previous iteration and a correction to it, which is to be determined. Let Φ_{inner} be the sum of these two contributions. The outer field is given according to Eq. (11) as the sum of the basic field Φ_0 and a correction ϕ. Φ_0 is defined by Eqs. (6), (7), and (10). The function $\overline{f}(\overline{\zeta})$ is given in parametric form in Eqs. (27) and (28) for $\alpha = 0$ and $\alpha = 1$, respectively. The position of the origin with respect to which ζ(via x,y(and z)) is defined may vary from iteration to iteration. The conditions (131) refer to

$$\phi = \Phi_{inner} - \Phi_0 \qquad (144)$$

In the application of Eqs. (131) one must therefore have a characterization of Φ_{inner} (and its gradient), for instance by the values of ϕ at a system of grid points, or in a finite element characterization. Besides one needs the function Φ_0 and the "weight functions" $\omega^{(m)}$.

Assume that the outer boundary S of the computed flow field is defined by the coordinates at a finite number of points. The functions $\overline{\omega}$ have the form $G(\zeta,s)\ \exp\ (st) = G(\zeta,s)\rho^s$. The function $G(\zeta,s)$ oscillates in the subsonic region with an amplitude which does not change much even if $s \to \infty$. If along the outer boundary of the computed region ρ changes considerably, then there is a danger that the functions $\overline{\omega}$ for large values of s are close to linear dependence. (For large values of s the functions $\overline{\omega}$ are then important only at those points where ρ is large.) It is therefore desirable to choose the outer surface of the computed flow field close to a surface ρ = const (see Figure 1).

The points (x,y), (x,r) or (x,y,z) used in the computation of the inner field are considered as fixed. A subscript 0 characterizes the coordinates of the current origin used for the representation of the outer flow field.

One computes from Eq. (7) for the chosen points of S

$$\zeta = (\gamma+1)^{-1/3}(x-x_0)(y-y_0)^{-4/5} \qquad \text{for } \alpha = 0$$

$$\zeta = (\gamma+1)^{-1/3}(x-x_0)\,\tilde{r}^{-4/7} \qquad \text{for } \alpha = 1 \tag{145}$$

where

$$\tilde{r} = [(y-y_0)^2 + (z-z_0)^2]^{1/2}$$

or in axisymmetric flow,

$$\tilde{r} = r \tag{146}$$

One has, furthermore

$$\theta = \operatorname{arctg}\frac{y-y_0}{\tilde{r}}$$

Next, one determines

$$\overline{\zeta} = \mu^{-1}\zeta$$

and then finds σ, either from a table, or by a solution of the second of Eqs. (27) or (28), for instance by means of a Newton iteration. This then allows one to determine $\overline{f}$ from the second of Eqs. (27) or (28) and $f = \mu^3\,\overline{f}$ from Eq. (10). Then one obtains Φ_0 from Eq. (6).

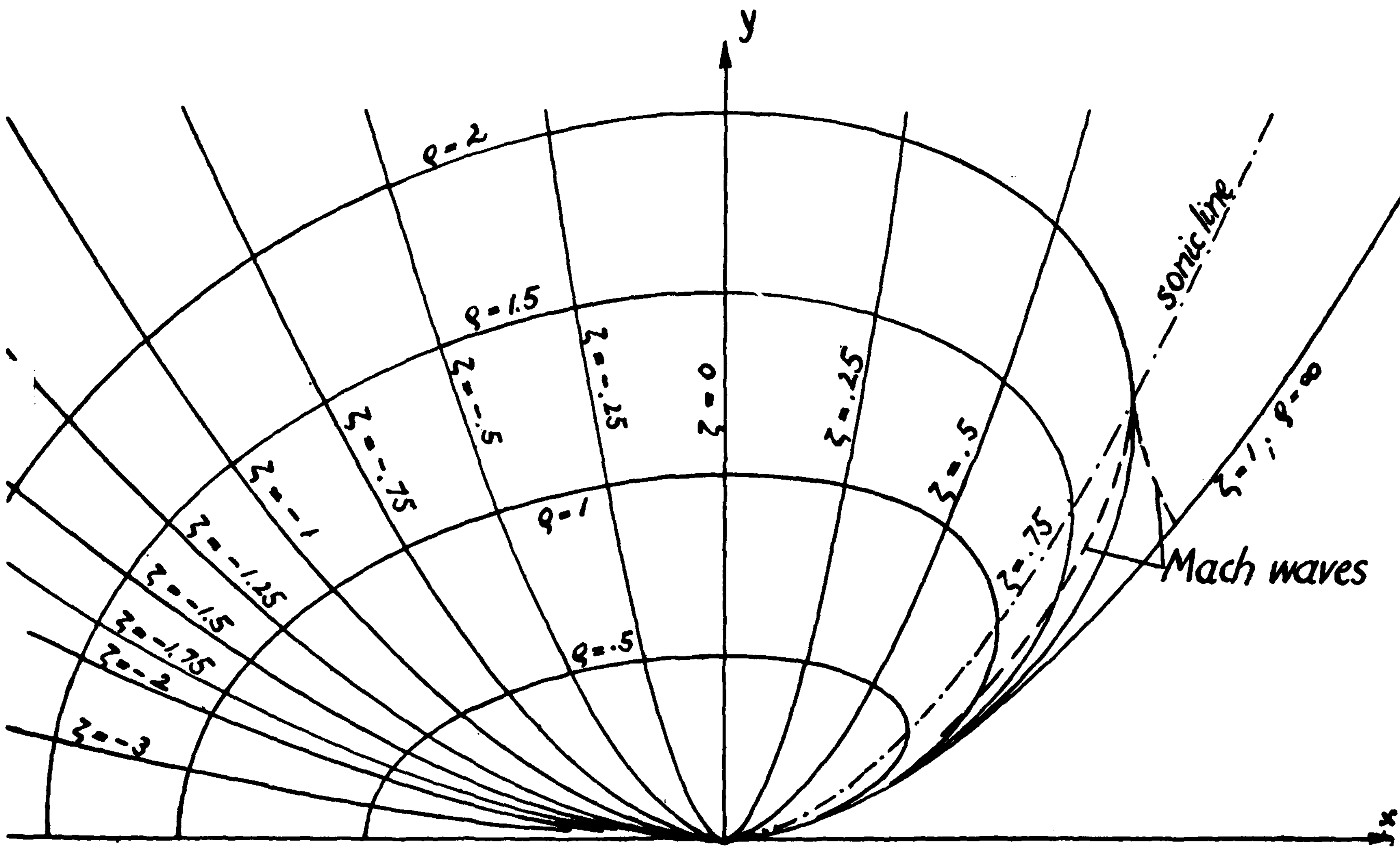

Figure 1. The ζ-ρ System for Axisymmetric Flows.

$$\Phi_0 = (y - y_0)^{2/5} f \qquad \text{for } \alpha = 0$$

$$\Phi_0 = \tilde{r}^{-2/7} f \qquad \text{for } \alpha = 1$$

These expressions are used in Eq. (144).

The integrals (131) will probably be evaluated in the form of Eqs. (40), (41), and (42) for, respectively, three-dimensional and plane flow in Cartesian coordinates, and three dimensional flow in cylindrical coordinates. Axisymmetric flows arise by specialization of Eq. (42). The coordinates used in these systems are those used for the computation of the inner flow field. The functions ω to be used are computed from Eq. (116) in the coordinates of the outer field. The functions $\bar{\omega}$ are defined in Eqs. (118). The function $\mathfrak{g}$ occurring in Eqs. (118) are defined in Eq. (77) in terms of σ. Here $\sigma_L = 16/3$ for $\alpha = 0$, and $\sigma_L = 6/5$ for $\alpha = 1$. The values of a, b, and c occurring in these formulae are found in Eq. (76), the value of R, which occurs here is given in Eq. (72). The values of s for which these functions ω are to be given for the plane case in Eqs. (139) and (140). For the three-dimensional case, they are computed from Eq. (82), with the provision taken from Eq. (13) that $Re(s) > -2/7$. The number of functions $\omega^{(m)}$ to be used equals the number of parameters which one uses to characterize the function ϕ at the outer boundary surface s.

The formulation of the conditions Eq. (131) for a sufficient number of functions $\omega^{(m)}$ gives as many linear relations of a global character between the ϕ, ϕ_x, and ϕ_y at the outer boundary as are necessary for the computation of the flow field.

After the values of ϕ, ϕ_t and ϕ_σ are found, one uses Eqs. (143) and (142) to find a readjusted basic field Φ_0 which is to be used in the next iteration step. The constants $c_{sym,2/5}$, $c_{sym,-2/5}$ are the residuals of $\psi^{(m)}(s,\sigma)$ at the respective poles. They are found from Eq. (124) with the definitions Eqs. (122) or (123).

The values of s needed for this adjustment are (2/5) and -2/5) for the symmetric solutions and (-3/5) for antisymmetric solutions in the plane case. For the case $\alpha = 1$, the values of s are taken from Eq. (90), they are $s = -2/7$ and $s = -6/7$ for $m = 0$, and $s = -9/7$ for $m = 1$.

In evaluating the integrals K, one must observe the behavior of the integrand at the limiting characteristic $\sigma = \sigma_L$, which is determined by the behavior of $\bar{\omega}$. One recognizes from Eq. (118) that for $\sigma = \sigma_L$ and y regular

$$\bar{\omega} \sim (\sigma_L - \sigma)^{(5s/3)-(1/6)} \qquad \text{for } \alpha = 0$$

$$\bar{\omega} \sim (\sigma_L - \sigma)^{(7s/5)+(2/5)} \qquad \text{for } \alpha = 1$$

One obtains, in particular

$$\left.\begin{array}{ll} \bar{\omega} \sim (\sigma_L - \sigma)^{1/2} & s = 2/5 \\ \bar{\omega} \sim (\sigma_L - \sigma)^{-5/6} & s = -2/5 \\ \bar{\omega} \sim (\sigma_L - \sigma)^{-7/6} & s = -3/5 \end{array}\right\} \qquad \alpha = 0$$

$$\left.\begin{array}{ll} \bar{\omega} \sim (\sigma_L - \sigma)^{0} & \text{for } s = -2/7 \\ \bar{\omega} \sim (\sigma_L - \sigma)^{-4/5} & \text{for } s = -6/7 \\ \bar{\omega} \sim (\sigma_L - \sigma)^{-7/5} & \text{for } s = -9/7 \end{array}\right\} \qquad \alpha = 1$$

Notice that in the integral K first derivatives of $\bar{\omega}$ are encountered. Except for the integrands for $s = 2/5$, $(\alpha = 0)$ for which the integral K is defined as an improper integral and for $s = -2/7$ for

$\alpha = 1$, for which no singularity is encountered, the integrals are defined by analytic continuation (Eq. (121)).

One needs for this purpose the first few terms of a power series development of ϕ and its gradient in terms of $(\sigma - \sigma_L)$ or an equivalent quantity. They must be extracted from the numerical data for ϕ. Only a few of these terms can be obtained with acceptable accuracy. The integral K is evaluated by one of the familiar procedures for $0 < \sigma < \sigma^1 \sigma_L$ and by means of formulae developed from Eq. (121) for the region $\sigma^1 < \sigma < \sigma_L$.

For the values of s which appear in Eq. (131) the integrals are integrable. It may, however, be worthwhile to examine to what extent the behavior of the function $\overline{\omega}$ at $\sigma = \sigma_L$ must be taken into account. These are the main considerations needed for a practical application.

SECTION IX
CONCLUDING REMARKS

The far field conditions (Eq. (131)) are global conditions; that is, they give relations connecting all points of the surface S. This makes their application in conjunction with an iterative procedure (as for instance the Murman Cole iterations) rather difficult. They are rather well suited for a correction to an existing field by means of a Newton-Raphson iteration in combination with a direct elimination procedure in the resulting linear equations. In such a procedure the sequence of the elimination of the unknown is important. One best arranges the problem in a manner that one obtains a large block multidiagonal matrix. The vector of the unknown is then partitioned, the first of the subvectors which arise in this manner express data at the surface of the body for which the flow is computed, the last of the subvectors gives data at the surface S. Usually the individual blocks are banded matrices, while the matrices derived from Eq. (131), which form the last row of blocks, are full. This does not matter because in the elimination process for the subvectors, one obtains full matrices rather quickly. These matrices have the same size as the matrices expressing Eq. (131). In any case, it seems desirable to use a coarser mesh at larger distances from the body.

The conditions Eqs. (131) are exact, provided that the linearization in the outer field used in Eq. (11) is applicable. The global character of these particular solutions is unavoidable. Reference 2 derives for another problem approximate conditions, valid for large distances, which have a local character. They would then be applicable for an iteration of the Murman-Cole type. The principal idea is to set up relations which are compatible with the dominant terms at infinity. The dominant term in the basic flow is

$$\phi = y^{2/5}\, f(\zeta) \qquad \text{for } \alpha = 0$$

$$\phi = r^{-2/7} f(\zeta) \qquad \text{for } \alpha = 1$$

Along a line ζ = const one has

$$\frac{d\phi}{dy} - \frac{2}{5}\frac{1}{y}\phi = 0 \qquad \text{for } \alpha = 0$$

$$\frac{d\phi}{dr} + \frac{2}{7}\frac{1}{y}\phi = 0 \qquad \text{for } \alpha = 1$$

These are localized for field conditions. The first perturbation occurs for $\alpha = 0$, at $s = 0$ according to Eq. (80). This leads to a boundary condition

$$\left(\frac{\partial}{\partial y} + \frac{1}{y}\right)\left(\frac{\partial}{\partial y} - \frac{2}{5}\frac{1}{y}\right)\phi = 0$$

(The derivatives are to be formed along lines ζ = const.) Indeed, if $\phi = y^{2/5}k_1(\zeta)$, then the condition is satisfied because of the second operator; if one substitutes $\phi = k_2(\zeta)$, then the second operator gives $-2/5(1/y)k_2(\zeta)$, which gives zero if the first operator is applied to it. Here $k_1(\zeta)$ and $k_2(\zeta)$ are arbitrary functions of ζ. The second derivative with respect to y must then be expressed by derivatives along the surface S using the partial differential equations for Φ.

In a similar manner, one can obtain improved conditions for the three-dimensional case. Conditions of this kind are, of course, rather attractive, and they are preferable to such assumptions as $\phi = a^*x$ (a sonic free jet) or $\phi_\omega = 0$ at a distant line which actually means a rigid wind tunnel wall.

The use of the rigorous conditions would allow one to reduce the size of the computer flow field. Whether this is worthwhile from the point of view of computational economy remains to be seen. The present article provides the background for making this comparison.

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